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HOW TO CUT A CAKE FAIRLY

L. E. DUBINS and E. H. SPANIER,* University of California, Berkeley

Steinhaus [16] has written about the problem of dividing an object (such as a cake) among a finite number of people so that each is satisfied that he has received his fair share, though each may have a different opinion as to which parts of the cake are most valuable. In that article he presented, among other things, the affirmative solution of Banach-Knaster to this problem. Their solution stimulated us to ask whether it may not sometimes be possible to divide the cake so that each person feels that he receives strictly more than his share. We found that such divisions do exist whenever two or more of the participants value some part of the cake differently. It then seemed natural to try to formulate a definition of one division being better than another and to find out whether there exist best or optimal divisions.

In this paper we observe that positive solutions to these problems can be based on results in the literature, and relations, some old, some perhaps new, between these problems and other topics in the mathematical literature will be pointed out.

In Part I we restate some, but not all, of the ideas contained in the article of Steinhaus [16]. We also present other well-known problems involving the division or partitioning of one or more objects. It is intended that this part of the paper be a complete unit requiring little technical knowledge and that it can be read profitably by a reader whether or not he goes on to the second, and more technical, part of the paper.

In Part II the mathematical details are presented. The results are not new and center around some results of Lyapunov [13] and generalizations of these.

I. DIVISION PROBLEMS

We begin with the problem of dividing a cake between two people so that each is satisfied that he gets at least half the cake. Whether or not the two people agree as to what constitutes half the cake makes no difference as it is always possible to divide the cake in the desired manner.

A simple and well-known method of effecting such a division is for "one to cut, the other to choose." The one who cuts does not have a chance of receiving more than half the cake according to his measure unless he is willing to take the risk of receiving less than half. On the other hand, the one who chooses may have the opportunity to get more than half the cake according to his measure without incurring any such risk. Thus in a certain sense the procedure gives an advantage to the one who chooses. Nevertheless, it is fair in the sense that each person can ensure that he receives at least half the cake according to his own tastes independently of what the other does.

* This paper was prepared while one of its authors was a Regular Postdoctoral N.S.F. Fellow and the other was supported in part by the National Science Foundation under grant number G-10700.

Steinhaus [16] asked whether a fair procedure could be found for dividing a cake among n participants for $n>2$. He found a solution for $n=3$, and Banach and Knaster showed that the solution for $n=2$ can be extended in an elegant and simple way to arbitrary n.

Their solution to the problem is, in essence, as follows. A knife is slowly moved at constant speed parallel to itself over the top of the cake. At each instant the knife is poised so that it could cut a unique slice of the cake. As time goes by the potential slice increases monotonely from nothing until it becomes the entire cake. The first person to indicate satisfaction with the slice then determined by the position of the knife receives that slice and is eliminated from further distribution of the cake. (If two or more participants simultaneously indicate satisfaction with the slice, it is given to any one of them.) The process is repeated with the other $n-1$ participants and with what remains of the cake.

To show this is a fair method of distribution suppose that a participant (call him A) adopts the strategy of indicating satisfaction at that position of the knife where the piece cut off has a value of $1/n$th of the total value of the cake according to his measure. Then, independent of the strategies of the other participants, even allowing for coalitions and duplicity, it is assured that either (i) A is given the first piece of the cake, which is precisely $1/n$th of the cake according to his taste, or (ii) A becomes one of the $n-1$ participants who have a share in the remains of the cake, which is worth at least $(n-1)/n$ of the original cake according to A's own evaluation. It is easy to see (by induction on n) that A possesses a strategy which ultimately yields him at least $1/n$th of the original cake according to his own evaluation.

Of course, the solution of Banach and Knaster shows that there exists a division of the cake into n pieces such that the jth piece is worth at least $1/n$th of the cake according to the jth measure. But their solution is more than a mere existence theorem. In fact, it provides an important practical method for effecting such a division; moreover it is a method which does not require the services of an umpire or an expert to decide what the "true" value of each piece of the cake really is. Another important property of their solution is that the piece that each participant receives is an "elementary" set in a sense that is easily made precise.

The method described above is equally applicable for the division of any object provided only that (1) the value assigned by any participant to any part of the object equals the sum of the values of the subparts when the part is subdivided into any finite number of subparts; and (2) the value to each participant of the potential slice varies in a continuous fashion as the knife is moved over the object. The method is also applicable when the participants are not to share equally but to share in some preassigned way as long as each one has been assigned a rational share of the cake. Thus, suppose α_i is a nonnegative rational number with $\sum\alpha_i=1$ for $1\leqq i\leqq n$. Then the method above can be applied

to effect a division in which the ith participant receives at least α_i of the total value of the object according to his evaluation. To see this, express each α_i as the ratio of two integers where the denominator is common to all i (so $\alpha_i = r_i/k$). Now make believe there are k participants, and let the real ith participant have a multiple personality and play the role of r_i of the fictitious participants. The method of division described above guarantees him $1/k$th of the cake in each of his roles. Therefore, he is assured of at least r_i/k of the cake according to his evaluation.

The constructive method for dividing a cake among n people described above gives each person a fair share according to his taste. However, it may result in a distribution for which one or more of the participants feels that someone else got more (or less) than his fair share. Steinhaus in [16] asserted that using methods similar to those of Stone and Tukey [17] it is possible to prove the existence of a partition of the cake into n sets so that each set has measure equal to $1/n$th of the total value of the cake to each participant (he actually asserted more, namely Corollary 1.1 below for the case $k=n$).

The problem considered above involves partitioning the cake into n sets and then evaluating each of n measures on each of the sets. We now describe another problem, a special case of which involves partitioning a set into k pieces and then evaluating each of n measures on each piece. The problem is called the "Problem of the Nile," and below we state a slightly modified version of Fisher's presentation of it [8, 9].

"Each year the Nile would flood, thereby irrigating or perhaps devastating parts of the agricultural land of a predynastic Egyptian village. The value of different portions of the land would depend upon the height of the flood. In question was the possibility of giving to each of the k residents a piece of land whose value would be $1/k$ of the total land value no matter what the height of the flood."

The problem as described above allows an infinite number of flood heights and in such a case, as shown by Feller [7], need not have a solution. Assuming that there are only a finite number, say n, of possible flood heights for the Nile, then Corollary 1.1 below (with $\alpha_j = 1/k$ for each j) shows the existence of a solution. That the problem has a solution under this hypothesis was first noted by Neyman [14]. Another contribution to this problem of the Nile is due to Tukey [19].

Closely related to the problem of the Nile is the "Problem of Similar Regions" of Neyman and Pearson [15]. Instead of partitioning a set, this problem involves for each α between 0 and 1, the existence of a single subset such that for each of n measures the ith measure of the subset is to the ith measure of the whole set as α is to 1. This problem, seemingly easier than the problem of the Nile, especially if α is rational, is actually equivalent to it as was first observed by Darmois [4].

The special case of the problem of similar regions in which the preassigned

ratio is equal to $\frac{1}{2}$ we shall call the "bisection problem." It is known (and we shall actually see in Part II) that a solution of the bisection problem leads to a solution of the problem of the Nile so that they are, in fact, equivalent.

Other problems involving bisection have been considered, including various forms of the "Ham Sandwich Problem" [17, 18]. This stimulating problem of Ulam, according to Tucker [18], got its name because Steinhaus picturesquely formulated it as the problem of cutting a ham sandwich composed of ham, butter, and bread into two parts by one slice of a knife in such a way that each ingredient is halved.* The general problem involves simultaneously cutting in half each of n bodies in euclidean n-space by a hyperplane. For our purposes we shall consider the following slightly more general problem. We suppose we are given n finite measures in euclidean n-dimensional space such that each measure vanishes on any set whose volume is zero, and we seek to find a hyperplane bisecting the space with respect to each measure. Lemma 5.1 of Part II is related to this problem, and the proof of this lemma can be modified slightly to solve this form of the ham sandwich problem.

In Part II it will be shown that a solution to the ham sandwich problem (or Lemma 5.1) implies the solution of the general bisection problem of any set. These two problems are not equivalent, however, because the ham sandwich problem involves not merely bisection but bisection in a special way, namely by a hyperplane.

In Part II the standard proof of the ham sandwich problem (and Lemma 5.1) using the Borsuk-Ulam theorem [3, 17, 18, 19] is presented. This theorem asserts

If f is a continuous map of the surface of the sphere in n-dimensional space into $(n-1)$-dimensional space such that $f(-x)=-f(x)$ for every x then there is some point on the sphere mapped into the origin.

Thus, this topological theorem, which does not involve division or measure in its statement, provides a logical starting point for the solution of the problems discussed above, though, historically, Ulam's ham sandwich problem seems to have come first.

II. EXISTENCE THEOREMS

1. Introduction. Let $u=u_1, \cdots, u_n$ be an n-tuple of countably additive finite real-valued functions defined on a σ-algebra $\mathfrak{U}$ of subsets of a set U. (Though the special case in which the u_i are probability measures is of primary interest here, it is convenient to treat the general case.) With every ordered partition P of U into k measurable sets $A_1, \cdots, A_k$, $A_j \in \mathfrak{U}$ for $1 \leqq j \leqq k$, we associate the $n \times k$ matrix of real numbers $M(P)=(u_i(A_j))$. Our main objective is to show that the range R of the matrix-valued function M is compact. The study of the case $k=2$ is essentially the same as the study of the range of vector-valued measures $u_1(A), \cdots, u_n(A)$ as A ranges over $\mathfrak{U}$. This case has been

* Interesting further developments were obtained by Steinhaus. See Fund. Math., vol. 33, 1945, pp. 245–263, with an historical footnote p. 255.

treated by Lyapunov [10, 13] who showed that R is compact. It is noteworthy that in order to prove closure he had to observe, and prove, that if each u_i is nonatomic then R is convex. A simplified proof of Lyapunov's results has been given by Halmos [10], and interesting extensions of Lyapunov's theorems were obtained by Blackwell [1, 2] and by Dvoretsky, Wald and Wolfovitz [5, 6] and Karlin [12]. Our principal result is a special case of theirs and is indeed explicitly formulated and proven in [6]. Our proof that R is convex if each u_i is nonatomic is based on the measure-algebra isomorphism theorem of Halmos and von Neumann [11] and on the Borsuk-Ulam theorem. The main theorem is:

THEOREM 1. *If u is nonatomic, then R is a compact convex set of matrices.*

This result will be proved in sections 5 and 6. We observe, however, that it follows from Theorems 3 and 5 of [2]. That is, identifying $n \times k$ matrices with nk-dimensional euclidean space, let A be the closed subset of this euclidean space composed of the k points $p_1, \cdots, p_k$, where p_j is the matrix having 1's in the jth column and 0's elsewhere. Consider the nk measures v_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$, where $v_{ij} = u_i$; then our set R is exactly the range considered by Blackwell (because a measurable function f from U to this set A corresponds to a measurable partition $P = A_1, \cdots, A_k$ such that $f(A_j) = p_i$).

We draw some consequences of this result for the cake problem. Let α_j be a set of n nonnegative numbers such that $\sum \alpha_j = 1$. We are interested in giving the ith person α_i of the cake in terms of the measure u_i. If $\alpha_i = 0$, the ith person has no share in the cake, and the number of participants is less than n. Therefore, there is no loss of generality in assuming each $\alpha_j > 0$. The following result shows that U can be partitioned so that each person believes the jth person receives α_j of U.

COROLLARY 1.1. *If each u_i is a nonatomic probability measure, then given k and $\alpha_1, \cdots, \alpha_k \geqq 0$ with $\sum \alpha_j = 1$, there exists a partition $A_1, \cdots, A_k$ of U such that $u_i(A_j) = \alpha_j$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, k$.*

Proof. Let P_j $(j = 1, \cdots, k)$ be the partition in which $A_j = U$ and A_r is empty if $r \neq j$. Then $M(P_j)$ is the matrix having 1's in the jth column and 0's everywhere else. Theorem 1 implies that $\sum \alpha_j M(P_j)$ is in R. Hence, there exists a partition P such that the jth column of $M(P)$ equals the jth column of $\sum \alpha_j M(P_j)$ and therefore P has all its entries equal to α_j.

This corollary confirms the assertion of Steinhaus. It also yields an affirmative answer to the problem of the Nile provided that there are only a finite number of flood heights.

The next result shows that each person can be given strictly more than his share of U (*i.e.*, the jth person receives more than α_j of U) provided that there are at least two people with different measures. A different proof of the result, which does not use the convexity theorem, has been shown to us by Jacob Feldman.

COROLLARY 1.2. *Suppose that each u_i is a nonatomic probability measure and $u_j \neq u_k$ for some $j \neq k$. Let $\alpha_i > 0$ with $\sum \alpha_i = 1$. Then there exists a partition $A_1, \cdots, A_n$ such that $u_i(A_i) > \alpha_i$ for each i.*

Proof. Suppose, for example, that u_1 and u_2 are not identical. Then for some measurable A, $u_1(A) > u_2(A)$. Let B be the complement of A. Then clearly $u_2(B) > u_1(B)$. Without loss of generality (by symmetry) we can suppose that $u_1(A)/\alpha_1 \geqq u_2(B)/\alpha_2$. Let P_0 be the ordered partition determined by giving A to 1, B to 2 and nothing to i if $i > 2$. For each $i > 1$ let P_i be the partition obtained by giving all of U to i and nothing to any other index. For each n-tuple x_i with $x_i \geqq 0$ and $\sum x_i = 1$ it follows from Theorem 1 that there is a partition $P = (A_1, \cdots, A_n)$ such that

$$M(P) = x_1 M(P_0) + \sum_{i \geq 2} x_i M(P_i).$$

Letting D denote the diagonal of $M(P)$ (so that the ith entry of D is $u_i(A_i)$) and letting D_i denote the diagonal of $M(P_i)$, we see that $D = x_1 D_0 + \sum_{i \geq 2} x_i D_i$. We shall try to choose the x_i so that all the entries of D are in the same ratios as the α_i. Hence, we want to solve the equations

$$x_1 u_1(A) = \lambda \alpha_1, \qquad x_1 u_2(B) + x_2 = \lambda \alpha_2, \qquad x_i = \lambda \alpha_i \qquad \text{for } i > 2.$$

Solving for x_i, summing, and using the fact that $\sum \alpha_i = 1$, we find that if we choose

$$\lambda = \left(1 + \frac{\alpha_1}{u_1(A)} [1 - u_1(A) - u_2(B)]\right)^{-1},$$

we have a solution. Now $u_1(A) + u_2(B) > 1$ so that $1 - u_1(A) - u_2(B) < 0$. Since $\alpha_1/u_1(A) < 1$, it follows that $\lambda > 1$. Therefore, choosing x_i to satisfy the above equations for this value of λ, we find that the ith entry of D is $\lambda \alpha_i > \alpha_i$ for all i, and this completes the proof.

The reader primarily interested in Lyapunov's theorem (or Theorem 1) will find little difficulty in skimming over Sections 2–4.

2. Preliminaries. We shall introduce hypotheses only when we have a need for them. We start with a set U, the object to be partitioned, and a finite sequence of finite real valued functions $u_1, \cdots, u_n$ each defined on the same nonempty collection $\mathfrak{U}$ of subsets of U. If $A \in \mathfrak{U}$, then $u_j(A)$ is the value or utility of the set A to the jth person. If P is the ordered partition $A_1, \cdots, A_n$ of U into sets $A_j \in U$ and the jth person is given the set A_j, then there are various criteria which might be applied to decide when one such partition is better than another. One of the simplest such criteria is to assign to the partition P the value $\sum u_j(A_j)$ and to define one partition to be better than another if it has a greater value. Interest now focuses on the existence of best (or maximal) partitions in this sense.

If there exists at least one partition and at most a finite number, it is trivial that a best one exists. Such is the case, for example, if U is a finite set and $\mathfrak{U}$ is the collection of all its subsets. When U is infinite, however, it is generally necessary to impose restrictions on the u_j. For example, if U is the set of positive integers, $\mathfrak{U}$ the collection of all its subsets, u_1 any countably additive probability measure on $\mathfrak{U}$ such that every integer has positive measure, and u_2 is a finitely additive function on $\mathfrak{U}$ vanishing on finite subsets and such that $u_2(U)=1$, then it is easy to see that no best partition exists. However, best partitions do exist if each utility function u_i is a countably additive measure as the following result shows.

THEOREM 2. *Let $\mathfrak{U}$ be a σ-algebra of subsets of U and let u_i, $1\leqq i\leqq n$ be a countably additive real-valued measure on $\mathfrak{U}$. Then the supremum of $\sum u_i(A_i)$, $1\leqq i\leqq n$ is attained as $P=A_1, \cdots, A_n$ ranges over ordered measurable partitions of U.*

Proof. Let v be any nonnegative finite-valued measure with respect to which each u_i is absolutely continuous (*e.g.*, v can be $\sum u_i$ if each u_i is nonnegative, or, more generally, $\sum|u_i|$, where $|u_i|$ is the total variation measure corresponding to u_i). Let f_i be a Radon-Nikodym derivative of u_i with respect to v (*i.e.*, f_i is a v-measurable function such that $u_i(A)=\int_A f_i dv$ for each $A\in\mathfrak{U}$). Let $f=\sup f_i$. We assert:

(i) For every P, $\sum u_i(A_i)\leqq\int f dv$.
(ii) There exists P such that $\sum u_i(A_i)=\int f dv$.

For (i) we observe that

$$\sum u_i(A_i)=\sum\int_{A_i} f_i dv\leqq\sum\int_{A_i} f dv=\int f dv.$$

For (ii) let A_i be the subset of U where $f_j<f$ for $j<i$ and $f_i=f$. Then A_i is measurable and clearly $A_1, \cdots, A_n$ is a partition of U. For this partition we have

$$\sum u_i(A_i)=\sum\int_{A_i} f_i dv=\sum\int_{A_i} f dv=\int f dv,$$

which completes the proof.

Let us return to the general case of any nonempty collection $\mathfrak{U}$ of subsets of a set U and real-valued functions $u_1, \cdots, u_n$ defined on $\mathfrak{U}$. We now discuss another notion of optimal partition somewhat more complicated than the above. The idea is to find a partition that maximizes the amount received by the person who gets the least, and, among such partitions, to find one that maximizes the amount received by the person who gets next to the least, etc. More precisely, for each partition $P=A_1, \cdots, A_n$ arrange the numbers $u_j(A_j)$ in nondecreasing order and call the resulting sequence

$$a_1(P) \leqq \cdots \leqq a_n(P).$$

P is called an *optimal* partition if for any partition P' either $a_i(P)=a_i(P')$ for all i, or if j is the smallest i such that $a_i(P)\neq a_i(P')$, then $a_j(P')<a_j(P)$ (*i.e.*, P is maximal in the ordering of partitions defined by lexicographic ordering of the sequences $a_1(P), \cdots, a_n(P)$). Thus if Q is the partition $B_1, \cdots, B_n$, then $\max_Q \min_i u_i(B_i)=a_1(P)$, and among all partitions Q such that $\min_j u_j(B_j)=a_1(P)$, then if $u_iB_i=a_1(P)$, $a_2(P)=\max_Q \min_{j\neq i} u_j(B_j)$, etc.

In a similar way if we were attempting to partition U according to the ratios α_i (*i.e.*, so that $u_i(A_i)\geqq\alpha_i$), where $\alpha_i>0$ and $\sum\alpha_i=1$, we let

$$a_1(P) \leqq \cdots \leqq a_n(P)$$

be the nondecreasing order of the numbers $u_i(A_i)/\alpha_i$, and then we order partitions lexicographically by these numbers. Thus optimal partitions in this sense would correspond to partitions P in which the smallest ratio $u_i(A_i)/\alpha_i$ was as large as possible, etc.

Clearly the compactness of the set of vectors $u_1(A_1), \cdots, u_n(A_n)$ implies the existence of optimal partitions. Thus attention is turned to this matter of compactness. The vector $u_1(A_1), \cdots, u_n(A_n)$ is the diagonal of the matrix $(u_i(A_j))$, where $1\leqq i, j\leqq n$. This matrix represents the amount that each person believes each participant receives in the partition $A_1, \cdots, A_n$. The study of this set of matrices is, therefore, of interest, and, in particular, the compactness of this set implies the compactness of the set of diagonals. More generally, for each positive integer k, let $R=R(k)$ be the collection of all $n\times k$ matrices $(u_i(A_j))$ as $A_1, \cdots, A_k$ ranges over all partitions of U into k sets $A_j\in\mathfrak{U}$. Our objective is to find conditions implying the compactness of R.

3. Compactness of R in certain special cases. To prove the compactness of R it would clearly suffice to find a topology on the set $\mathcal{P}$ of k-partitions $P=A_1, \cdots, A_k$ such that (i) $\mathcal{P}$ is compact, and (ii) if $M(P)=(u_i(A_j))$, then M is a continuous matrix-valued function on $\mathcal{P}$. Since $\mathcal{P}$ is a subset of $(\mathfrak{U})^k$, the k-fold cartesian product of $\mathfrak{U}$, it would suffice to find a topology on $\mathfrak{U}$ such that (1) $\mathfrak{U}$ is compact, (2) $\mathcal{P}$ is a closed subset of $(\mathfrak{U})^k$, and (3) u is a continuous vector-valued function on $\mathfrak{U}$.

When $\mathfrak{U}$ is the collection of all subsets of U, there is a natural compact topology to put on $\mathfrak{U}$. In fact, the set of all subsets of U is in 1-1 correspondence with the cartesian power over U of a 2-point space, and this has a compact topology under the product topology. In this topology a directed family A_j of subsets of U converges if and only if $\limsup A_j=\liminf A_j$ (where, as usual, $\limsup A_j=\bigcap_j \bigcup_{j'>j} A_{j'}$, $\liminf A_j=\bigcup_j \bigcap_{j'>j} A_{j'}$). Since the maps $A, B\to A\cup B$ and $A, B\to A\cap B$ are both jointly continuous in this topology, it follows that $\mathcal{P}$ is a closed subset of $(\mathfrak{U})^k$. Therefore, this topology on $\mathfrak{U}$ satisfies (1) and (2) above, and to verify (3) we need only show that u is continuous on $\mathfrak{U}$. This is a reasonable condition to impose in the case U is countable, but if U is nondenumerable

this is too restrictive a condition. A better one is to ask that whenever a *sequence* of sets A_p converges to a a set A, then $u(A_p)$ should converge to $u(A)$. In the event U is countable and $\mathfrak{U}$ is the collection of all its subsets this condition is equivalent to continuity because in this case $\mathfrak{U}$ is a compact metric space and so the topology is completely described by its convergent sequences. Therefore, we have proved:

THEOREM 3. *Let $u=u_1, \cdots, u_n$ be defined on the collection $\mathfrak{U}$ of all subsets of a countable set U with values in n-dimensional euclidean space. Suppose that whenever $A_p \to A$ in U then $u(A_p) \to u(A)$. Then the collection of matrices $(u_i(A_j))$, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$, as $A_1, \cdots, A_k$ ranges over all partitions of U is a compact set.*

When U is uncountable and $\mathfrak{U}$ is a σ-algebra of subsets of U, we do not know if the above sequential condition on u is enough to imply the compactness of R. Indeed, this is unknown even if $\mathfrak{U}$ is the collection of all subsets of U. In fact we do not know whether there are any functions u other than the function identically zero defined on all subsets of U satisfying the condition $A_p \to A$ implies $u(A_p) \to u(A)$ and vanishing on countable A. If u is also assumed to be finitely additive (and, therefore, countably additive) Ulam [20] has shown the nonexistence of a nontrivial u for sets U of appropriate cardinality.

4. The purely atomic case. Henceforth we shall suppose that u is a countably additive finite real-valued measure defined on a σ-algebra $\mathfrak{U}$ of subsets of U. A measurable set A is said to be an *atom* for u if $u(A) \neq 0$ and for every measurable set B either $u(A \cap B) = 0$ or $u(A - B) = 0$. Clearly if A is an atom for u then for each i it is also an atom for u_i if $u_i(A) \neq 0$. It is worth noting that the converse does not hold. For example, let U consist of two distinct points a_1 and a_2, let $\mathfrak{U}$ consist of the four subsets of U, and let

$$u_i(a_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then U is an atom for both u_1 and u_2, but it is not an atom for u.

A measure u is said to be *purely atomic* if there exists a disjoint collection K of atoms for u with the property that for each measurable set B, $u(B) = \sum u(B \cap A)$, $A \in K$. u is said to be *nonatomic* provided that there are no atoms for u. We observed above that an atom for each u_i need not be an atom for u; however, it is true that u is purely atomic if and only if each u_i is. We also have the following result:

LEMMA 4.1. *u is nonatomic if and only if each u_i is nonatomic.*

Proof. We saw above that an atom for u is an atom for at least one u_i. Therefore, if all the u_i are nonatomic, so is u.

To prove the converse we prove by induction that if A is an atom for the vector measure $v_k = u_1, \cdots, u_k$, then there exists an atom for the vector measure $v_{k+1} = u_1, \cdots, u_{k+1}$. A is the union of A' and A'', where u_{k+1} is nonnegative

on all subsets of A' and nonpositive on all subsets of A''. Either A' or A'' is again an atom for v_k. Without loss of generality, therefore, we can assume that we have an atom, say A, for v_k such that u_{k+1} is nonnegative on all subsets of A.

Consider the collection of subsets B of A which are atoms for v_k. This collection is easily seen to be closed under finite intersections, and therefore under countable intersections. Hence, there exists such a subset B_0 such that $u_{k+1}(B_0) \leqq u_{k+1}(B)$ for all such B. An easy verification shows that B_0 is an atom for v_{k+1} so the induction is complete, and the lemma is proved.

Given a measure u let K be a maximal disjoint collection of atoms for u (which exists by Zorn's lemma). It is trivial that K is at most denumerable so its union U_1 is measurable. Clearly u restricted to U_1 is purely atomic and restricted to $U - U_1$ is nonatomic. Hence, any measure u splits into a purely atomic part and a nonatomic part whose sum equals u.

If u is purely atomic, it is easy to verify that it is isomorphic to a measure defined on the set of all subsets of some countable set. Therefore, Theorem 3 can be applied and we obtain:

THEOREM 4. *Let u be a purely atomic n-dimensional vector measure. Then R, the range of the $n \times k$ matrix valued function of k-partitions, is compact.*

This result disposes of the purely atomic case, and we now pass on to the consideration of the more interesting nonatomic case.

5. Lyapunov's convexity theorem. The proof of Lyapunov's theorem presented here uses the following two results. (For a proof of the first see [3], [18], p. 293 while the second is proved in [11], Theorem C, p. 173.)

Borsuk-Ulam Theorem. Let f be a continuous mapping of the n-dimensional sphere S^n into n-dimensional euclidean space such that $f(x) + f(-x) = 0$ for every $x \in S^n$. Then there exists $x \in S^n$ such that $f(x) = 0$.

Halmos-Von Neumann Isomorphism Theorem. Any two separable nonatomic measure algebras of countably additive probability spaces are isomorphic.

The convexity theorem we are after asserts that the range R of our $n \times k$ matrix-valued function of partitions is convex if u is nonatomic. This is a corollary of Lyapunov's theorem which asserts that the range of a nonatomic vector-valued measure is convex. We present a sequence of lemmas leading to these results.

LEMMA 5.1. *Let $\mathcal{S}$ be the σ-algebra of Borel subsets of S^n, let w be the usual rotation invariant measure on $\mathcal{S}$, and let $w_1, \cdots, w_n$ be n countably additive real-valued measures on $\mathcal{S}$ each absolutely continuous with respect to w. Then there exists a closed hemisphere E of S^n such that for all i, $w_i(E) = \frac{1}{2} w_i(S^n)$.*

Proof. For each $x \in S^n$ let $E(x)$ be the closed hemisphere of all $y \in S^n$ whose inner product with x is nonnegative. Then $E(x) \cup E(-x) = S^n$ and

$w(E(x)\cap E(-x))=0$. If x and x' are close points of S^n, then the symmetric difference of $E(x)$ and $E(x')$ has small w-measure. Since each w_i is absolutely continuous with respect to w, it also has small w_i measure. Therefore, for each i, the function $w_i(E(x))$ is a continuous function on S^n. Define f on S^n by

$$f(x) = (w_1(E(x)) - \tfrac{1}{2}w_1(S^n), \cdots, w_n(E(x)) - \tfrac{1}{2}w_n(S^n)).$$

Then f is a continuous map of S^n into euclidean n-space such that $f(x)+f(-x)=0$ for all $x\in S^n$. The Borsuk-Ulam theorem implies the existence of $x\in S^n$ such that $f(x)=0$. Then $E(x)$ satisfies the condition that $w_i(E(x))=\frac{1}{2}w_i(S^n)$ for all i.

COROLLARY. *Let $E_1, \cdots, E_n$ be measurable subsets of S^n. Then there exists a hemisphere that contains half of each E_i.*

We observe in passing that the techniques used in [17] also yield the following slightly generalized version of the ham sandwich theorem: *Given any n countably additive finite real-valued measures defined on the Borel subsets of n-dimensional euclidean space that are absolutely continuous with respect to Lebesgue measure, there exists a hyperplane that bisects each measure.*

Using the Halmos-Von Neumann isomorphism theorem we now show that Lemma 5.1 implies the solution of the bisection problem in any space.

The structure of this section is represented schematically by the following diagram showing the implications to be established and the number of the lemma in which the implication in question is proved.

$$\text{Borsuk-Ulam Theorem} \overset{(5.1)}{\Longrightarrow} \text{Ham Sandwich Theorem}$$

$$\overset{(5.2)}{\Longrightarrow} \text{General Bisection Theorem}$$

$$\overset{(5.3)}{\Longrightarrow} \text{Existence of monotone one-parameter family of subsets}$$

$$\overset{(\text{Theorem 5})}{\Longrightarrow} \text{Lyapunov Convexity Theorem} \overset{(5.4)}{\Longrightarrow} \text{Matrix convexity.}$$

LEMMA 5.2. *Let u be a nonatomic countably additive vector-valued measure defined on a σ-algebra $\mathfrak{U}$ of subsets of U. Then there exists a set A of $\mathfrak{U}$ such that $u(A)=\frac{1}{2}u(U)$.*

Proof. Let $v=\sum|u_i|$, where $|u_i|$ is the total variation of the ith component of u. Then by (4.1), v is a nonatomic positive measure. It can be verified that either the measure ring of v is separable or else it contains a nonatomic separable subalgebra. Let v' denote the restriction of v to such a separable subalgebra. By the Halmos-Von Neumann theorem the measure ring of v' is isomorphic to the measure ring $\mathfrak{S}$ on S^n so that v' corresponds to the measure w. Each u_i has a restriction u_i' to the subalgebra which corresponds to a measure w_i on $\mathfrak{S}$, and since u_i is absolutely continuous with respect to v so is w_i with respect to w. By (5.1) there is a closed hemisphere E of S^n such that $w_i(E)=\frac{1}{2}w_i(S^n)$ for each i. Let A

be the element of $\mathfrak{U}$ corresponding to E (under the isomorphism of the separable subalgebra with $\mathfrak{s}$). Then $u_i(A) = \frac{1}{2}u_i(U)$ for all i, completing the proof.

The next result shows that the solution of the bisection problem can be used to find a one-parameter monotone family of subsets $A(t)$ such that $u(A(t)) = tu(U)$.

LEMMA 5.3. *Let u be a nonatomic vector measure as in* (5.2). *There exists a function $t \to A(t)$ mapping the closed unit interval $[0, 1]$ into $\mathfrak{U}$ such that*:

(1) $u(A(t)) = tu(U)$, (2) $t_1 < t_2 \Rightarrow A(t_1) \subset A(t_2)$.

Proof. Let Q be the collection of functions $t \to A(t)$ defined on subsets of $[0, 1]$ satisfying (1) and (2) where defined. We define a partial ordering in Q by saying one function is larger than another if its domain of definition is larger and its restriction to the smaller domain equals the smaller function. Zorn's lemma can be applied to Q to prove the existence of a maximal function A. We need only show A is defined on all of $[0, 1]$. Let D be the domain of A. Then if t_i is a monotone sequence in D converging from below (or from above) to a t not in D, we can extend A to $D \cup \{t\}$ by defining $A(t) = \cup A(t_i)$ (or $A(t) = \cap A(t_i)$). This contradicts the maximality of A. Therefore, monotone sequences in D converge in D so that D is closed.

Now suppose D is not all of $[0, 1]$. Since 0 and 1 are clearly in D (because if not we could extend A by defining $A(0) = \varnothing$ and $A(1) = U$) and D is closed, there exist $t_1, t_2 \in D$ with $t_1 < t_2$ such that no point of the open interval (t_1, t_2) is in D. By (2) we have $A(t_1) \subset A(t_2)$. The restriction of u to the measurable subsets of $A(t_2) - A(t_1)$ satisfies the hypothesis of (5.2) so there exists a measurable subset B of $A(t_2) - A(t_1)$ such that $u(B) = \frac{1}{2}u(A(t_2) - A(t_1))$. We can extend A to $D \cup \{\frac{1}{2}t_1 + \frac{1}{2}t_2\}$ by defining $A(\frac{1}{2}t_1 + \frac{1}{2}t_2) = B \cup A(t_1)$. This contradicts the maximality and therefore $D = [0, 1]$.

It should be noticed that Lemma 5.3 is equivalent to a generalization of an interesting theorem discovered by Neyman [14]. *Given any n nonatomic countably additive finite real-valued measures u_i, $i = 1, \cdots, n$ defined on a σ-algebra $\mathfrak{U}$ of subsets of a set U, there exists a measurable function f such that $u_i[f^{-1}(-\infty, t)] = tu_i(U)$.*

We now show that (5.3) yields a proof of Lyapunov's convexity theorem.

THEOREM 5 (Lyapunov). *The range of a nonatomic vector-valued measure is convex.*

Proof. Let A_1, A_2 be two elements of $\mathfrak{U}$ and let $t_1, t_2 \geqq 0$ with $t_1 + t_2 = 1$. Consider the $2n$-dimensional vector-valued measure m defined on $\mathfrak{U}$ by $m(B) = (u(B \cap A_1), u(B \cap A_2))$. Lemma 5.3 implies the existence of a set B such that $m(B) = t_1 m(U)$ or, equivalently,

$$u(B \cap A_1) = t_1 u(A_1), \qquad u(B \cap A_2) = t_1 u(A_2).$$

Therefore, letting B^c = complement of B in U, we have

$$u(B^c \cap A_2) = u(A_2) - u(B \cap A_2) = (1 - t_1)u(A_2) = t_2u(A_2).$$

Let $A = (B \cap A_1) \cup (B^c \cap A_2)$. Then

$$u(A) = u(B \cap A_1) + u(B^c \cap A_2) = t_1u(A_1) + t_2u(A_2),$$

completing the proof.

COROLLARY 5.4. *For any positive integer k if u is a nonatomic vector-valued measure then the range $R(k)$ of the $n \times k$ matrix-valued function M of k-partitions $P = A_1, \cdots, A_k$ defined by $M(P) = (u_i(A_j))$ is convex.*

Proof. Let $A_1, \cdots, A_k$ and $A_1', \cdots, A_k'$ be two partitions of U and let $0 < t < 1$. Let m be the $2n \times k$ vector-valued measure determined by the real-valued measures $B \to (u_i(B \cap A_j), u_i(B \cap A_j'))$ as B ranges over $\mathfrak{U}$. By Theorem 5 there exists a set B such that $u_i(B \cap A_j) = tu_i(A_j)$ and $u_i(B \cap A_j') = tu_i(A_j')$. Therefore,

$$u_i(B \cap A_j) = tu_i(A_j), \qquad u_i(B^c \cap A_j') = (1 - t)u_i(A_j').$$

It follows that if we set $E_j = (B \cap A_j) \cup (B^c \cap A_j')$, then

$$u_i(E_j) = tu_i(A_j) + (1 - t)u_i(A_j').$$

The simple verification that $E_1, \cdots, E_k$ forms a partition of U completes the proof. (The device of stringing together a number of vector-valued measures is borrowed from Blackwell [2].)

6. Compactness of R. Lyapunov proved not only that the range of a vector valued measure is convex but also that it is compact. Though the convexity of R was shown to be a simple corollary to Lyapunov's convexity theorem via Blackwell's device, the compactness of R does not follow easily, if at all, from Lyapunov's compactness theorem. Therefore a proof that R is compact will be given here that has the same general pattern as Halmos' presentation of Lyapunov's compactness theorem [10].

Consider first the nonatomic case. We know that R is bounded in the nk-dimensional euclidean space E, and convex by (5.4). For any set R to be closed it of course suffices that $\overline{R} - R$ be empty (where $\overline{R}$ denotes the closure of R). For any convex set R, and any $y \in \overline{R} - R$, there is a supporting hyperplane H of $\overline{R}$ not containing all of R with $y \in H$. To see this let E' be the affine variety generated by R. A standard separation theorem for convex sets implies the existence of a hyperplane of support H' of $\overline{R}$ in E' that contains y. Let H be any hyperplane of E whose intersection with E' is H'. Then H is the desired hyperplane, because $y \in H$, and, since H' is a proper subvariety of E', there is an $r \in R$ not in H'. Certainly $r \in E'$. But $H \cap E' = H'$, so $r \notin H$.

Returning to the general program, it now follows that to show $\overline{R} - R$ is empty, it suffices to show that if H is a supporting hyperplane of $\overline{R}$ not containing all of R, then $\overline{R} \cap H \subset R$. Therefore the proof that R is compact is reduced to establishing the following two propositions

(6.1) $$\overline{R} \cap H \subset \overline{R \cap H},$$

(6.2) $R \cap H$ is closed.

The hyperplane H is associated with a linear function t and a constant c such that

(i) $t(x) \leqq c$ for all $x \in R$, (ii) $t(x) = c$ for all $x \in H$,

(iii) for all $\epsilon > 0$ there exists $x \in R$ such that $t(x) > c - \epsilon$.

Consider the following proposition which implies (6.1)

(6.3) *Given $\epsilon > 0$ there exists $\delta > 0$ such that for $x \in R$ with $t(x) > c - \delta$, there exists $x^* \in R$ such that $t(x^*) = c$ and $|x_{ij} - x^*_{ij}| < \epsilon$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq k$.*

Proof that (6.3)$\Rightarrow$(6.1). Let $y \in \overline{R} \cap H$. To prove (6.1) we show that every 2ϵ-neighborhood of y meets $R \cap H$. Choose $\delta > 0$ according to (6.3). The set of x such that $t(x) > c - \delta$ is open and contains y. Since $y \in \overline{R}$, the intersection of this set with the ϵ-neighborhood of y meets R, so there is $x \in R$ such that $t(x) > c - \delta$ and $|y_{ij} - x_{ij}| < \epsilon$ for all i, j. Applying (6.3) we obtain a point $x^* \in R \cap H$ such that $|x_{ij} - x^*_{ij}| < \epsilon$ for all i, j. Then $|y_{ij} - x^*_{ij}| < 2\epsilon$ for all i, j, and the proof is complete.

We now concentrate on the proof of assertion (6.3) The linear function t is associated to an array t_{ij} of real numbers such that $t(x) = \sum t_{ij} x_{ij}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$. Let $u'_j = \sum_i t_{ij} u_j$.

LEMMA 6.4. *The number c is the least upper bound of $\sum u'_j(A_j)$ as $A_1, \cdots, A_k$ ranges over all measurable partitions of U into k sets.*

Proof. If $x = M(P)$, where $P = A_1, \cdots, A_k$, then $t(x) = \sum_{i,j} t_{ij} u_i(A_j) = \sum_j u'_j(A_j)$. Since $c = \sup_{x \in R} t(x)$, the result follows.

Let $v = \sum |u_i|$ where $|u_i|$ is the total variation measure corresponding to u_i. Let f_j be a Radon-Nikodym derivative of u'_j with respect to v (which exists because u'_j is obviously absolutely continuous with respect to v). Let f be the supremum of the k functions $f_1, \cdots, f_k$ and σ an arbitrary nonempty subset of the k integers $1, \cdots, k$. Define F_σ to be the set of $z \in U$ such that $f_i(z) = f(z)$ for $i \in \sigma$ and $f_j(z) < f(z)$ for $j \notin \sigma$. It is clear that each F_σ is a measurable set, that any two are disjoint, and that their union is U; hence, they form a measurable partition of U. Furthermore, if i and i' are in σ, then the restriction of u'_i to F_σ is identical with the restriction of $u'_{i'}$ to F_σ, and we shall denote this restriction by u_σ.

LEMMA 6.5. $\sum u_\sigma(F_\sigma) = c$.

Proof. Given any partition $A_1, \cdots, A_k$ of U we have

$$(6.6) \qquad \sum_j u_j'(A_j) = \sum_{j,\sigma} u_j'(F_\sigma \cap A_j) \leqq \sum_{j,\sigma} u_\sigma(F_\sigma \cap A_j) = \sum_\sigma u_\sigma(F_\sigma).$$

This shows that $\sum u_\sigma(F_\sigma)$ is an upper bound for $\sum_j u_j'(A_j)$. If, however, we define a partition $A_1, \cdots, A_k$ by the condition that A_j is the union of those F_σ for which j is the smallest integer in σ, then we see that the inequality in (6.6) above is an equality and $\sum u_\sigma(F_\sigma) = \sum u_j'(A_j)$ for this partition. Therefore, $\sum u_\sigma(F_\sigma)$ is the least upper bound of the numbers $\sum u_j'(A_j)$ so, by (6.4), equals c.

LEMMA 6.7. *Suppose $j \notin \sigma$ and that A is a measurable subset of F_σ such that $v(A) > 0$. Then $u_\sigma(A) > u_j'(A)$.*

Proof. If $i \in \sigma$, we have

$$u_\sigma(A) - u_j'(A) = \int_A f_i dv - \int_A f_j dv = \int_A (f_i - f_j) dv.$$

But $f_i - f_j > 0$ on F_σ and hence on A. Since $v(A) > 0$ and v is countably additive, the right-hand side of the above equation is positive and the proof is complete.

LEMMA 6.8. *$\sum u_j'(A_j) = c$ if and only if for all σ and all $j \notin \sigma$, $v(F_\sigma \cap A_j) = 0$.*

Proof. $\sum u_j'(A_j) = c$ if and only if equality holds in (6.6). This is equivalent to

$$u_j'(F_\sigma \cap A_j) = u_\sigma(F_\sigma \cap A_j)$$

for all j, σ. The latter holds automatically if $j \in \sigma$, and for $j \notin \sigma$ is equivalent to $v(F_\sigma \cap A_j) = 0$, by (6.7), which completes the proof.

LEMMA 6.9. *Given $\epsilon > 0$ there exists $\delta > 0$ such that if $A_1, \cdots, A_k$ is a measurable partition with $\sum u_j'(A_j) > c - \delta$ then there exists a partition $B_1, \cdots, B_k$ such that:*

$$(1) \qquad \sum u_j'(B_j) = c, \qquad (2) \qquad v((A_j - B_j) \cup (B_j - A_j)) < \epsilon \text{ for all } j.$$

Proof. Lemma 6.7 shows that if $j \notin \sigma$ then, on F_σ, v is absolutely continuous with respect to $u_\sigma - u_j'$. Therefore, given $\epsilon > 0$ there exists δ such that for all pairs j, σ with $j \notin \sigma$, if $A \subset F_\sigma$ and $(u_\sigma - u_j')(A) < \delta$, then $v(A) < \epsilon/\bar{k}$ (where $\bar{k}$ is the number of pairs (j, σ) with $j \notin \sigma$). We show this δ satisfies the conditions of the lemma.

Let $A_1, \cdots, A_k$ be a partition with $\sum u_j'(A_j) > c - \delta$. Consider the partition $A_j \cap F_\sigma$ for all j, σ. We have

$$\sum_{j,\sigma} u_j'(A_j \cap F_\sigma) = \sum_j u_j'(A_j) > c - \delta = \sum_{j,\sigma} u_\sigma(A_j \cap F_\sigma) - \delta.$$

Therefore, $\sum_{j,\sigma} [u_\sigma(A_j \cap F_\sigma) - u_j'(A_j \cap F_\sigma)] < \delta$. Since each term of the sum is nonnegative, it follows that $u_\sigma(A_j \cap F_\sigma) - u_j'(A_j \cap F_\sigma) < \delta$ for all j, σ. Therefore, if $j \notin \sigma$ it follows from the choice of δ that $v(A_j \cap F_\sigma) < \epsilon/\bar{k}$.

For each $j \notin \sigma$ let $i(j, \sigma)$ be the smallest integer in σ. We define a partition $B_1, \cdots, B_k$ by

$$B_i = \bigcup \{A_i \cap F_\sigma \mid i \in \sigma\} \cup \bigcup \{A_j \cap F_\sigma \mid j \notin \sigma \text{ and } i = i(j, \sigma)\}.$$

Then $B_i \subset \bigcup_{i \in \sigma} F_\sigma$ so if $i \notin \sigma$, $B_i \cap F_\sigma$ is empty, and by (6.8), $\sum u_j'(B_j) = c$, so the partition $B_1, \cdots, B_k$ satisfies (1). To see that it satisfies (2) we have

$$(A_i - B_i) \cup (B_i - A_i) \subset \bigcup_{j \in \sigma} A_j \cap F_\sigma,$$

and therefore

$$v((A_i - B_i) \cup (B_i - A_i)) \leqq \sum_{j \in \sigma} v(A_j \cap F_\sigma) < \bar{k} \cdot \epsilon / \bar{k} = \epsilon,$$

and this completes the proof.

Clearly (6.3) is an immediate consequence of (6.9) so it only remains to verify (6.2) to prove the compactness of R. The proof of compactness proceeds by induction on d = the dimension of R. If $d = 0$, R is a single point and there is nothing to prove. Now we assume inductively that for all n, k whenever we have a nonatomic n-dimensional vector-valued measure on a measure space such that the dimension of the $n \times k$ matrix-valued function of k-partitions has dimension $< d$ then it is compact. Assume dim $R = d$. We show the inductive assumption implies that $R \cap H$ is closed for every supporting plane H not containing R (*i.e.*, it implies (6.2)), and then, having (6.1), we obtain the compactness of R, and the induction is complete.

It only remains, therefore, to prove $R \cap H$ is closed using the inductive hypothesis. Note that $\dim(R \cap H)$ is necessarily less than d because H does not contain R. We use the same notation as earlier in this section.

For each σ let R_σ be the set of all $n \times k$ matrices of the form $u_i(C_j)$, where $C_1, \cdots, C_k$ is a partition of F_σ such that $v(C_j) = 0$ if $j \notin \sigma$. Given such a partition $C_j(\sigma)$ of F_σ for every σ, we define a partition $A_1, \cdots, A_k$ of U by setting $A_j = \bigcup_\sigma C_j(\sigma)$. Then (6.8) shows that $(u_i(A_j))$ is a matrix in $R \cap H$. Conversely, if $(u_i(A_j))$ is a matrix of $R \cap H$, (6.8) shows that for each σ, $A_j \cap F_\sigma = C_j(\sigma)$ is a partition of F_σ into sets $C_1(\sigma), \cdots, C_k(\sigma)$ such that $v(C_j(\sigma)) = 0$ if $j \notin \sigma$. The relation between the partition A_j and the partitions $C_j(\sigma)$ is such that $(u_i(A_j)) = \sum_\sigma (u_i(C_j(\sigma)))$. Therefore, $R \cap H = \sum_\sigma R_\sigma$. Since $\sum_\sigma R_\sigma$ lies in $R \cap H$, which has dimension $< d$, each R_σ has dimension less than d. By the inductive assumption each R_σ is compact (if $k(\sigma)$ is the number of elements of σ, then R_σ is clearly homeomorphic to the range of the $n \times k(\sigma)$ matrix-valued function defined on F_σ by partitions into $k(\sigma)$ sets and by the measures u_i). Therefore, $R \cap H$ is compact, and this completes the proof of compactness in the nonatomic case. This together with Theorem 5 completes the proof of Theorem 1.

Theorem 4 asserted the compactness of R in the purely atomic case. Combining this with the result just proved and using the decomposition of an arbitrary measure into its purely atomic and nonatomic parts as described in Section 4 we have the following.

THEOREM 6. *The range R of the $n \times k$ matrix-valued function M of partitions $P = A_1, \cdots, A_k$ defined by $M(P) = (u_i(A_j))$ is compact.*

Returning to the concept of optimal partition introduced in Section 2 we have the following.

COROLLARY 6.10. *There exist optimal partitions.*

We conclude with one further notion. An optimal partition $A_1, \cdots, A_n$ will be called *equitable* if $u_i(A_i) = u_j(A_j)$ for all i and j. In the event that the measures u_j are not only nonatomic but also each is absolutely continuous with respect to every other, then every optimal partition is equitable. Even in the nonatomic case the converse does not hold.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION*

L. E. BUSH, Kent State University

THE EIGHTEENTH ANNUAL COMPETITION, FEBRUARY 8, 1958

Problems. Part I

1. If $a_0, a_1, \cdots, a_n$ are real numbers satisfying

$$\frac{a_0}{1}+\frac{a_1}{2}+\cdots+\frac{a_n}{n+1}=0,$$

show that the equation $a_0+a_1x+a_2x^2+\cdots+a_nx^n=0$ has at least one real root.

2. Two uniform solid spheres of equal radii are so placed that one is directly above the other. The bottom sphere is fixed, and the top sphere, initially at rest, rolls off. At what point will contact between the two spheres be "lost"? Assume the coefficient of fraction is such that no slipping occurs.

3. Real numbers are chosen at random from the interval $(0 \leqq x \leqq 1)$. If after choosing the nth number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e.

4. If $a_1, a_2, \cdots, a_n$ are complex numbers such that

$$|a_1| = |a_2| = \cdots = |a_n| = r \neq 0,$$

and if ${}_nT_s$ denotes the sum of all products of these n numbers taken s at a time, prove that $\left|{}_nT_s/{}_nT_{n-s}\right| = r^{2s-n}$ whenever the denominator of the left-hand side is different from zero.

5. Show that the integral equation

$$f(x, y) = 1 + \int_0^x \int_0^y f(u, v)\,du\,dv$$

has at most one solution continuous for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

6. What is the smallest amount that may be invested at interest rate i, compounded annually, in order that one may withdraw 1 dollar at the end of the first year, 4 dollars at the end of the second year, $\cdots$, n^2 dollars at the end of the nth year, in perpetuity?

7. Show that ten equal-sized squares cannot be placed on a plane in such a way that no two have an interior point in common and the first touches each of the others.

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* Reprints will be available about April 1, 1961, from Professor H. M. Gehman, Executive Director, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y. at 25¢ for single copies and 20¢ each for orders of five or more.

Problems. Part II

1. a) Given line segments A, B, C, D, with A the longest, construct a quadrilateral with these sides and with A and B parallel, when possible.
   b) Given any acute angled triangle ABC and one altitude AH, select any point D on AH, then draw BD and extend until it intersects AC in E. Draw CD and extend until it intersects AB in F. Prove angle AHE = angle AHF.
2. Prove that the product of 4 consecutive positive integers cannot be a perfect square or cube.
3. In a round-robin tournament with n players (each pair of players plays one game) in which there are no draws, the numbers of wins scored by the players are $s_1, s_2, \cdots, s_n$. Prove that a necessary and sufficient condition for the existence of 3 players A, B, C, such that A beat B, B beat C and C beat A is

$$s_1^2 + s_2^2 + \cdots + s_n^2 < (n-1)(n)(2n-1)/6.$$

4. What is the average straight line distance between two points on a sphere of radius 1?
5. Given an infinite number of points in a plane, prove that if all the distances determined between them are integers then the points are all in a straight line.
6. A projectile moves in a resisting medium. The resisting force is a function of the velocity and is directed along the velocity vector. The equation $x = f(t)$ gives the horizontal distance in terms of the time t. Show that the vertical distance y is given by

$$y = -gf(t)\int \frac{dt}{f'(t)} + g\int \frac{f(t)}{f'(t)}\,dt + Af(t) + B,$$

   where A and B are constants and g is the acceleration due to gravity.
7. Prove that if $f(x)$ is continuous for $a \leqq x \leqq b$ and $\int_a^b x^n f(x)dx = 0$ for $n = 0, 1, 2, \cdots$ then $f(x)$ is identically zero on $a \leqq x \leqq b$.

Solutions.* Part I

1. Let $f(x) = \sum_{i=0}^{n} (a_i x^{i+1})/(i+1)$. Then $f(0) = f(1) = 0$ and, by the mean value theorem, $f'(\theta) = 0$ for some real θ such that $0 < \theta < 1$. Hence $\sum_{i=0}^{n} a_i x^i$ has a real root.

2. Let M be the mass and r the radius of the rolling sphere. Taking the center of the stationary sphere as origin, let $R(\theta)$ be the position vector of the center of

* The solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

The solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. The published solutions are intended to be mere outlines. In order to be considered correct in a contestant's paper, more detail would have to be given.

the rolling sphere with θ as the angle between $R(\theta)$ and the upward vertical through the origin. Clearly $R(\theta)$ remains in a vertical plane as the motion takes place. The centrifugal force is $2Mr\dot{\theta}^2$, where $\dot{\theta}=d\theta/dt$. The component of the gravitational force which is directed toward the origin is $Mg\cos\theta$, and the spheres separate when $2rM\dot{\theta}^2$ just exceeds $Mg\cos\theta$, *i.e.*, the moment of separation occurs when $2r\dot{\theta}^2=g\cos\theta$. The loss of potential energy is $2rMg(1-\cos\theta)$ and must equal the total kinetic energy which is the translational energy $2Mr^2\dot{\theta}^2$ plus the rotational energy $\frac{2}{5}Mr^2\dot{\theta}^2$. Thus $\dot{\theta}^2=5g(1-\cos\theta)/(7r)$ and the separation occurs when $\cos\theta=10/17$.

3. Let $P_n(x)=\Pr(\sum_{i=1}^{n}x_i<x)$. Then

$$P_{n+1}(x)=\int_0^x(x-y)dP_n(y)=\int_0^x P_n(y)dy.$$

Since $P_1(x)=x$, it follows that $P_n(x)=x^n/n!$. The probability that the sum first exeeds 1 on the $(n+1)$th trial is

$$Q_{n+1}=\int_0^1 ydP_n(y)=\int_0^1\frac{y^n}{(n-1)!}dy=\frac{n}{(n+1)!}.$$

Hence $\bar{n}=\sum_{n=2}^{\infty}nQ_n=\sum_{n=2}^{\infty}1/(n-2)!=e$.

4. Let $a_j=r\exp(\theta_j)$, $j=1,\cdots,n$. Then

$${}_nT_s=r^s\sum_{C_s}\exp\left(i(\theta_{j_1}+\cdots+\theta_{j_s})\right)$$

and the summation is over C_s, the set of all combinations of $1,\cdots,n$ taken s at a time. Similarly,

$${}_nT_{n-s}=r^{n-s}\sum_{C_{n-s}}\exp\left(i(\theta_{j_1}+\cdots+\theta_{j_{n-s}})\right).$$

Let $\Omega=\sum_{j=1}^{n}\theta_j$. Then

$${}_nT_{n-s}=r^{n-s}\sum_{C_s}\exp\left(i(\Omega-\theta_{j_1}-\cdots-\theta_{j_s})\right),$$

$$\left|{}_nT_{n-s}\right|=r^{n-s}\left|\sum_{C_s}\exp\left(i(\theta_{j_1}+\cdots+\theta_{j_s})\right)\right|=r^{n-2s}\left|{}_nT_s\right|.$$

5. The difference of two solutions, say h, must satisfy the homogeneous equation $h(x,y)=\int_0^x\int_0^y h(u,v)dudv$. Let $M_p=\max h(x,y)$ for $0\leqq x,y\leqq p<1$. By the mean value theorem for integrals and the continuity of h, $M_p=h(x_p,y_p)\leqq M_p x_p y_p\leqq M_p p^2$. Thus $M_p\leqq 0$ for all p such that $0<p<1$. Hence $h(x,y)\leqq 0$ for $0\leqq x,y\leqq 1$ by the continuity of h. By a symmetrical argument, $h(x,y)\geqq 0$ and it follows that $h(x,y)=0$.

6. The present value of n^2 dollars n years from now at rate i per year is $n^2(1+i)^{-n}$. Thus the required sum is $\sum_{n=1}^{\infty}n^2(1+i)^{-n}$. Since $(1-x)^{-1}=\sum_{n=0}^{\infty}x^n$, it follows by differentiation that

$$x(1-x)^{-2}=\sum_{n=1}^{\infty} nx^n, \qquad (x+x^2)(1-x)^{-3}=\sum_{n=1}^{\infty} n^2x^n,$$

all these series being convergent for $-1<x<1$. Taking $x=(1+i)^{-1}$, the required sum is found to be $(1+i)(2+i)i^{-3}$.

7. Let Q be a square of side 1 and let Q^* be the square of side 2 having common center with Q and sides parallel to the corresponding sides of Q. The proof follows by showing that any square P of side 1 having only boundary points in common with Q must contain a portion of the perimeter of Q^* of length 1 or greater. The various cases can be classified as: (1) A corner of P is on a side of Q at a distance l from a nearest corner of Q and the smallest acute angle between the sides of Q and P is θ. (2) A side of P touches a corner of Q, the point of contact being distant l from the nearest vertex of P, and the acute angle between the sides of Q and P is θ. Each of (1) and (2) can be separated into two subcases according as a vertex of Q^* lies in P or not. All other cases are limiting cases of these. The discussion in each case is elementary.

Solutions. Part II

1. a) Construct $A-B$ and a triangle with sides $A-B$, C, and D, if this is possible. The condition for this is that any one of these segments be less than the sum of the other two. The quadrilateral is readily completed by extending side $A-B$ by a segment of length B and constructing a parallel to the nearer side of the triangle.

b) If one chooses a cartesian coordinate system with H as origin and with HA and CB as axes, the slopes of HE and HF are readily computed and one is the negative of the other.

2. Since $x(x+1)(x+2)(x+3)=(x^2+3x+1)^2-1$, the product of four consecutive positive integers differs by 1 from a perfect square and hence is not a square. Of four consecutive positive integers, x, $x+1$, $x+2$, $x+3$, either $x+1$ or $x+2$ is prime to the product of the other three. Thus, if the product of four consecutive positive integers is a perfect cube, one of $x+1$ or $x+2$ is a perfect cube also. Thus the product of the remaining three positive integers is a perfect cube. But this is absurd since $x(x+2)(x+3)$ and $x(x+1)(x+3)$ lie strictly between $(x+1)^3$ and $(x+2)^3$ if $x>1$. The case $x=1$ is tested directly.

3. If A beats B, B beats C, and C beats A, we shall say that these players form a "triangle." If two players have the same score, say A and B have score k, then a triangle must occur. Since A beats B or B beats A, $k>0$. Say that A beats B so that B beats k players other than A and B, and A only beats $k-1$ players other than A and B. Thus B beats some player who beats A. Conversely, if all scores are distinct, they must be $0, 1, \cdots, n-1$ and clearly no triangles are possible. Thus the scores $s_1, \cdots, s_n$ are $0, 1, \cdots, n-1$ if and only if no triangles occur. Also, $\sum_{i=1}^{n} s_i^2 \leqq \frac{1}{6}(n-1)(n)(2n-1)$, with equality only when the scores in order are $s_i=i-1$. This is true for $n=2, 3$ by inspection. Assuming the truth of the assertion for $n-1$ players, the result follows if $s_n=n-1$ and some

other $s_i \neq i-1$. If $s_n < n-1$, then some $s_i > i-1$. If s_n is increased by 1 and s_i decreased by 1, the sum of squares is increased by $2(s_n - s_i)+2 > 0$. By a second finite induction, the case $s_n = n-1$ is reached and the result follows.

4. By symmetry it suffices to determine the average length of chords emanating from a single point, say the North Pole. Using spherical coordinates (with $\rho = 1$), the areal element may be taken as $2\pi \sin\phi d\phi$, where ϕ is the angle which the radius to the North Pole makes with the radius to the other end of the chord. The integral required is $(4\pi)^{-1}\int_0^\pi (2 \sin \tfrac{1}{2}\phi)(2\pi \sin\phi)d\phi = 4/3$.

5. Assume that P_1, P_2, P_3 are not collinear and are in the set. Any point P_4 in the set lies on the line P_iP_j or on one of the hyperbolas

$$\left| d(P, P_i) - d(P, P_j) \right| = 1, 2, \cdots, d(P_i, P_j) - 1,$$

where $i \neq j$, $i, j = 1, 2, 3$. Since any two such loci have at most four intersections, the possible positions for P_4 are finite in number.

6. Since $mx''(t) = Rx'(t)$ and $my''(t) = -mg + Ry'(t)$, it follows that $R = m\{f''(t)/f'(t)\}$. Hence $\{y'(t)/f'(t)\}' = -g/f'(t)$ and

$$y'(t) = \{y'(0)/f'(0)\}f'(t) - gf'(t)\int_0^t \{1/f'(\tau)\}d\tau.$$

Integrating once more,

$$y(t) = y(0) + \{y'(0)/f'(0)\}\{f(t) - f(0)\} - g\int_0^t f'(x)dx \int_0^x \{1/f'(\tau)\}d\tau.$$

7. Evidently $\int_a^b P(x)f(x)dx = 0$ for any polynomial $P(x)$. Given $\epsilon > 0$, $P(x)$ can be selected so that $|f(x) - P(x)| < \epsilon$, $a \leqq x \leqq b$. Hence

$$\int_a^b f^2(x)dx = \left| \int_a^b f(x)\{f(x) - P(x)\}dx \right| \leqq \epsilon(b-a)M,$$

where M is a bound for $|f(x)|$ on $a \leqq x \leqq b$. This implies that $\int_a^b f^2(x)dx = 0$ and since f is continuous it must vanish identically on the interval.

THE NINETEENTH ANNUAL COMPETITION, NOVEMBER 22, 1958

Problems. Part I

1. Let $f(m, 1) = f(1, n) = 1$ for $m \geqq 1$, $n \geqq 1$, and let $f(m, n) = f(m-1, n) + f(m, n-1) + f(m-1, n-1)$ for $m > 1$ and $n > 1$. Also let

$$S(n) = \sum_{a+b=n} f(a, b), \qquad a \geqq 1 \text{ and } b \geqq 1.$$

Prove that $S(n+2) = S(n) + 2S(n+1)$ for $n \geqq 2$.

2. Let $R_1 = 1$, $R_{n+1} = 1 + n/R_n$, $n \geqq 1$. Show that for $n \geqq 1$, $\sqrt{n} \leqq R_n \leqq \sqrt{n+1}$.

3. Under the assumption that the following set of relations has a unique solution for $u(t)$, determine it.

$$\frac{du(t)}{dt} = u(t) + \int_0^1 u(s)ds, \qquad u(0) = 1.$$

4. In assigning dormitory rooms, a college gives preference to pairs of students in this order:

$$AA,\ AB,\ AC,\ BB,\ BC,\ AD,\ CC,\ BD,\ CD,\ DD$$

in which AA means two seniors, AB means a senior and a junior, etc. Determine numerical values to assign to A, B, C, D so that the set of numbers $A+A$, $A+B$, $A+C$, $B+B$, etc., corresponding to the order above will be in descending magnitude. Find the general solution and the solution in least positive integers.

5. Show that the number of nonzero terms in the expansion of the nth order determinant having zeros in the main diagonal and ones elsewhere is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right].$$

6. Let $a(x)$ and $b(x)$ be continuous functions on $0 \leqq x \leqq 1$ and let $0 \leqq a(x) \leqq a < 1$ on that range. Under what other conditions (if any) is the solution of the equation for u, $u = \max_{0 \leq x \leq 1} [b(x) + a(x) \cdot u]$, given by

$$u = \max_{0 \leq x \leq 1} \left[\frac{b(x)}{1 - a(x)}\right]?$$

7. Let a and b be relatively prime positive integers, b even. For each positive integer q let $p = p(q)$ be chosen so that

$$\left|\frac{p}{q} - \frac{a}{b}\right|$$

is a minimum. Prove that

$$\lim_{n \to \infty} \sum_{q=1}^{n} \frac{q\left|\frac{p}{q} - \frac{a}{b}\right|}{n} = \frac{1}{4}.$$

Problems. Part II

1. Given

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^{-1}, \qquad n \geqq 1.$$

Prove that

$$b_n = \frac{n+1}{2n} b_{n-1} + 1,\ n \geqq 2, \text{ and hence, as a corollary, } \lim_{n \to \infty} b_n = 2.$$

2. Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.
3. If a square of unit side be partitioned into two sets, then the diameter (least upper bound of the distances between pairs of points) of one of the sets is not less than $\frac{1}{2}\sqrt{5}$. Show also that no larger number will do.
4. Let C be a real number, and let f be a function such that

$$\lim_{x\to\infty} f(x) = C, \qquad \lim_{x\to\infty} f'''(x) = 0.$$

Prove that

$$\lim_{x\to\infty} f'(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f''(x) = 0,$$

where superscripts denote derivatives.
5. The lengths of successive segments of a broken line are represented by the successive terms of the harmonic progression $1, 1/2, \cdots, 1/n, \cdots$. Each segment makes with the preceding segment a given angle θ. What is the distance and what is the direction of the limiting point (if there is one) from the initial point of the first segment?
6. Let a complete oriented graph on n points be given, *i.e.*, a set of n points $1, 2, \cdots, n$, and between any two points i and j a direction, $i\to j$. Show that there exists a permutation of the points, $[a_1, a_2, \cdots, a_n]$, such that $a_1\to a_2 \to \cdots \to a_n$.
7. Let $a_1, \cdots, a_n$ be a permutation of the integers $1, \cdots, n$. Call a_i a "big" integer if $a_i > a_j$ for all $j>i$. Find the mean number of "big" integers over all permutations on the first n positive integers.

Solutions. Part I

1. $S(n+2)=\sum_{j=1}^{n+1} f(j, n+2-j)$

$$=f(1, n+1)+\sum_{j=2}^{n}\{f(j-1,n+2-j)+f(j,n+1-j)+f(j-1,n+1-j)\}+f(n+1,1)$$

$$=\left[f(n, 1)+\sum_{j=1}^{n-1} f(j,n+1-j)\right]+\left[\sum_{j=2}^{n} f(j,n+1-j)+f(1,n)\right]+\sum_{j=1}^{n-1} f(j, n-j)$$

$$=2S(n+1)+S(n).$$

2. For $n=1$, $\sqrt{n}\leqq R_n\leqq 1+\sqrt{n}$. Assume that the inequalities hold for $n=m\geqq 1$. Then

$$R_{m+1} = 1+\frac{m}{R_m} \geqq 1+\frac{m}{1+\sqrt{m}} = \frac{1+m+\sqrt{m}}{1+\sqrt{m}} > \sqrt{(m+1)}$$

since $(1+m+\sqrt{m})^2>(1+m)(1+\sqrt{m})^2=(1+m)^2+2(1+m)\sqrt{m}$. Also R_{m+1}

$\leqq 1+m/\sqrt{m}<1+\sqrt{(m+1)}$. Thus, by induction, the inequality is valid for all positive integers n.

3. Let $\int_0^1 u(t)dt=a$. Then $du/dt=u(t)+a$ and hence $u=-a+ce^t$, where c is a constant. Then $a=\int_0^1 u(t)dt=-a+c(e-1)$ and thus $2a=c(e-1)$. Since $u(0)=-a+c=1$, the function u must be defined by $u(t)=1+2(e^t-1)/(3-e)$.

4. By assumption, $2A>A+B>A+C>2B>B+C>A+D>2C>B+D>C+D>2D$. It is clear that $A>B>C>D$ and we may set $C=D+h$, $B=C+k=D+h+k$; $A=B+m=D+h+k+m$. Since $2C>B+D$ and $A+C>2B$, we must have $h>k$, $m>k$. Thus $m=k+n$, $h=k+p$ and $A=D+3k+n+p$, $B=D+2k+p$, $C=D+k+p$ are necessary. Since $B+C>A+D>2C$, we must have $3k+2p>3k+n+p>2k+2p$ or $p>n$ and $k+n>p$. Taking $p=n+q$ we must have $k>q$ or $k=q+r$. Thus $A=D+4q+3r+2n$, $B=D+3q+2r+n$, $C=D+2q+n+r$ are necessary. These are readily found to be sufficient by substitution. Since all the quantities are to be positive integers, the minimum solution is found by taking $D=r=n=q=1$. This solution is $A=10$, $B=7$, $C=5$, $D=1$.

5. Let N_n be the number of nonzero terms in the expansion of the given nth order determinant. Then N_n is equal to the number of permutations $(j_1, \cdots, j_n)$ of the positive integers $1, \cdots, n$. Let $(j_1, \cdots, j_{n+1})$ be any permutation of the positive integers $1, \cdots, n+1$, $j_i \neq i$, $i=1, \cdots, n$. Suppose that $n+1$ occurs in the rth position. Interchanging $n+1$ and j_{n+1} results in another permutation of $1, \cdots, n+1$. Two cases arise according as $j_{n+1}=r$ or $j_{n+1}\neq r$. In the first case the positive integers form a permutation of $n-1$ objects with each object out of its natural position, and the number of such permutations is N_{n-1}. As r varies from 1 to n there is thus a total of nN_{n-1} possible permutations in the first case. In the second case, there are N_n permutations for a fixed r and thus nN_n permutations formed in this way. Thus $N_{n+1}=n(N_n+N_{n-1})$ and since $N_2=1$, $N_3=2$, an easy induction establishes that $N_n=n!\sum_{k=2}^{n}(-1)^k/k!$

6. If $u=\max[b(x)+ua(x)]$, $0\leqq x\leqq 1$, then $u\geqq b(x)+ua(x)$ for all x in the unit interval and equality holds for some x in the range. Since the inequality is equivalent to $u\geqq b(x)\{1-a(x)\}^{-1}$, it follows that $u=\max[b(x)\{1-a(x)\}^{-1}]$ for $0\leqq x\leqq 1$. Thus the solution is unique. Conversely, if $u=\max[b(x)\{1-a(x)\}^{-1}]$ for $0\leqq x\leqq 1$, then $u\geqq b(x)\{1-a(x)\}^{-1}$ for all x in the range, equality holding for one such x. Hence $u=\max[b(x)+ua(x)]$, $0\leqq x\leqq 1$.

7. It is clear that p must be chosen to minimize $|bp-aq|$ and that if $q_1\equiv q_2 \pmod b$, then the corresponding minima are equal. Thus we consider $q=1, \cdots, b$. Since aq is congruent (mod b) to one of the integers $-\frac{1}{2}b, 1-\frac{1}{2}b, \cdots, \frac{1}{2}b$, it follows that the minimizing value of p is such that $bp-aq$ belongs to the above class. Moreover, if $q_2=q_1+r$, $0<r<b$, then the minimizing values of p are such that $bp_1-aq_1\neq bp_2-aq_2$ since otherwise $ar=b(p_2-p_1)$ and this is impossible since a must divide p_2-p_1. Thus as q varies from 1 to b, inclusive, the minimizing value of p is such that it varies over the reduced residue class (mod b) exactly once, and $|bp-aq|$ takes on the values $1, \cdots, -1+\frac{1}{2}b$ twice and the values 0 and $\frac{1}{2}b$ just once. Thus $\sum_{q=1}^{b}|bp-aq|=\frac{1}{4}b^2$. If $n=kb+r$, $0\leqq r<b$, then

$$(*) \qquad \frac{1}{nb}\sum_{q=1}^{n} |\, bp - aq \,| = \frac{kb^2}{4b(kb+r)} + \frac{C_n}{nb},$$

where $C_n = \sum_{q=kb+1}^{n} |\, bp - aq \,| \leqq \frac{1}{2}b^2$. As $n \to \infty$ and hence $k \to \infty$, it is clear that the right member of (*) approaches $\frac{1}{4}$.

Solutions. Part II

1. By definition,

$$b_n = \sum_{h=0}^{n} \frac{h!(n-h)!}{n!} = 1 + \sum_{h=0}^{n-1} \frac{h!(n-1-h)!(n-h)}{n!}$$

$$= 1 + \sum_{h=0}^{n-1} \frac{(n-1-h)!h!(h+1)}{n!}.$$

Adding the last two expressions and multiplying by n, one obtains

$$2nb_n = 2n + \sum_{h=0}^{n-1} \frac{h!(n-1-h)!(n+1)}{(n-1)!},$$

whence $b_n = 1 + \{(n+1)/(2n)\}b_{n-1}$. This may be written

$$b_n - 2 = \frac{n+1}{2n}(b_{n-1} - 2) + \frac{1}{n}.$$

Since $b_5 = 2 + 9/15$ and for $n \geqq 5$, $2 \leqq b_{n-1} \leqq 2 + 4/(n-1)$ implies that $2 \leqq b_n \leqq 2 + 4/n$, it follows that $\lim_{n\to\infty} b_n = 2$.

2. The statement is true for $n = 1$ and we assume it true for n. Let k_i, $i = 1, \cdots, n+2$, be positive integers such that $k_i < k_{i+1} \leqq 2n+2$, $i = 1, \cdots, n+1$, and suppose that no one of them divides another. If $k_{n+2} = 2n+2$, we may replace k_{n+2} by $n+1$ and the new set has the property that no element divides any other. Thus we may suppose $k_{n+2} < 2n+2$. But then the set k_i, $i = 1, \cdots, n+1$ contains no term exceeding $2n$ and thus contradicts the induction hypothesis.

3. Let A, B, C, D be consecutive vertices of the square. If F is the midpoint of segment AB and G the midpoint of segment CD, the partition with boundary FG shows that no larger number than $\frac{1}{2}\sqrt{5}$ is possible for the statement to be true. Now let S and T be the sets of any partitioning of the square. If opposite vertices belong to one of the sets, then the diameter of that set is $\sqrt{2} > \frac{1}{2}\sqrt{5}$. If this does not occur, we may assume, by relabeling vertices if necessary, that A and D belong to S and that C and D belong to T. Since F must be in S or in T, it follows that the diameter of one of these sets is at least $\frac{1}{2}\sqrt{5}$.

4. Given a positive real number a, we have

$$f(a \pm 1) = f(a) \pm f'(a) + \tfrac{1}{2}f''(a) \pm \tfrac{1}{6}f'''(a_{\pm}),$$

where $a < a_+ < a+1$, $a-1 < a_- < a$. Hence

$$f(a+1)-f(a-1)=2f'(a)+\tfrac{1}{6}f'''(a_+)+\tfrac{1}{6}f'''(a_-)$$
$$f(a+1)+f(a-1)=2f(a)+f''(a)+\tfrac{1}{6}f'''(a_+)-\tfrac{1}{6}f'''(a_-).$$

The result follows by taking limits as $a\to\infty$.

5. We may take the initial vector to be represented by the complex number 1 and then the nth vector is $n^{-1}\exp\{i\theta(n-1)\}$. If $r=\exp(i\theta)$, the sum of the first n vectors is $\sum_{j=0}^{n-1}\{r^j/(j+1)\}$. For $\theta\neq 0$, the corresponding infinite series converges by the Dirichlet test, and by Abel's theorem the sum is $-r^{-1}\log(1-r)$.

6. The result is trivial for $n=2$. An ordering such that $a_1\to a_2\to\cdots\to a_n$ will be called a chain. Assume that the result is true for $n<k$ and consider the case $n=k$. Let I be the set of those objects a_i such that $a_i\to a_k$ and let T be the set of those a_i such that $a_k\to a_i$. Both I and T may be ordered to form a chain, trivially if they have a single element or are empty and by the induction hypothesis in the contrary case. The elements of I, ordered as a chain followed by a_k and followed by the elements of T ordered as a chain, yield the chain for all elements.

7. Let S_n be the total number of "big" integers obtained from all the permutations of the positive integers $1,\cdots,n$. It is clear that S_n does not depend on the specific values but only on the ordering. Thus S_{n-1} is also the total number of "big" integers obtained from all permutations of the positive integers $2,\cdots,n$. Each such permutation gives rise to n permutations of $1,\cdots,n$ by inserting 1 in the n possible positions. Each permutation of $1,\cdots,n$ is obtained exactly once in this way. When 1 is annexed as the last element, the number of "big" numbers in that permutation is increased by 1, but when 1 is inserted in any of the other positions, the number of "big" numbers is not changed. Hence $S_n=nS_{n-1}+(n-1)!$ or

$$\frac{S_n}{n!}=\frac{S_{n-1}}{(n-1)!}+\frac{1}{n}.$$

Thus the average A_n of the "big" numbers satisfies the relation $A_n=A_{n-1}+n^{-1}$. Since $A_1=1$, it follows that $A_n=\sum_{j=1}^{n}j^{-1}$.

THE TWENTIETH ANNUAL COMPETITION, NOVEMBER 21, 1959

Problems. Part I

1. Let n be a positive integer. Prove that $x^n-(1/x^n)$ is expressible as a polynomial in $x-(1/x)$ with real coefficients if and only if n is odd.
2. Prove that if the points in the complex plane corresponding to two distinct complex numbers z_1 and z_2 are two vertices of an equilateral triangle, then the third vertex corresponds to $-\omega z_1-\omega^2 z_2$, where ω is an imaginary cube root of unity.
3. Find all complex-valued functions f of a complex variable such that $f(z)+zf(1-z)=1+z$ for all z.

4. If f and g are real-valued functions of one real variable, show that there exist numbers x and y such that $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and $|xy - f(x) - g(y)| \geqq \frac{1}{4}$.
5. A sparrow, flying horizontally in a straight line, is 50 feet directly below an eagle and 100 feet directly above a hawk. Both hawk and eagle fly directly toward the sparrow, reaching it simultaneously. The hawk flies twice as fast as the sparrow. How far does each bird fly? At what rate does the eagle fly?
6. Let m and n be integers greater than 1, and $a_1, \cdots, a_{m+1}$ real numbers. Prove that there exist real n by n matrices $A_1, \cdots, A_m$ such that (i) Det $(A_j) = a_j$ for $j = 1, \cdots, m$ and (ii) Det $(A_1 + \cdots + A_m) = a_{m+1}$.
7. If f is a real-valued function of one real variable which has a continuous derivative on the closed interval $[a, b]$ and for which there is no $x \in [a, b]$ such that $f(x) = f'(x) = 0$, then prove that there is a function g with continuous first derivative on $[a, b]$ such that $fg' - f'g$ is positive on $[a, b]$.

Problems. Part II

1. Let each of m distinct points on the positive part of the X-axis be joined to n distinct points on the positive part of the Y-axis. Obtain a formula for the number of intersection points of these segments (exclusive of end points), assuming that no three of the segments are concurrent.
2. Let c be a positive real number. Prove that c can be expressed in infinitely many ways as a sum of infinitely many distinct terms selected from the sequence

$$1/10, 1/20, \cdots, 1/10n, \cdots.$$

3. Give an example of a continuous real-valued function f from $[0, 1]$ to $[0, 1]$ which takes on every value in $[0, 1]$ an infinite number of times.
4. Given the following matrix of 25 elements

$$\begin{pmatrix} 11 & 17 & 25 & 19 & 16 \\ 24 & 10 & 13 & 15 & 3 \\ 12 & 5 & 14 & 2 & 18 \\ 23 & 4 & 1 & 8 & 22 \\ 6 & 20 & 7 & 21 & 9 \end{pmatrix},$$

choose five of these elements, no two coming from the same row or column, in such a way that the minimum of these five elements is as large as possible. Prove that your answer is correct.
5. Find the equation of the smallest sphere which is tangent to both of the lines: (i) $x = t + 1$, $y = 2t + 4$, $z = -3t + 5$, and (ii) $x = 4t - 12$, $y = -t + 8$, $z = t + 17$.
6. Prove that if x and y are positive irrationals such that $1/x + 1/y = 1$, then the sequences $[x], [2x], \cdots, [nx], \cdots$ and $[y], [2y], \cdots, [ny], \cdots$ together include every positive integer exactly once. (The notation $[x]$ means the largest integer not exceeding x.)

7. For each positive integer n, let f_n be a real-valued symmetric function of n real variables. Suppose that for all n and for all real numbers $x_1, \cdots, x_{n+1}, y$, it is true that

1) $f_n(x_1 + y, \cdots, x_n + y) = f_n(x_1, \cdots, x_n) + y$,

2) $f_n(-x_1, \cdots, -x_n) = -f_n(x_1, \cdots, x_n)$,

3) $f_{n+1}(f_n(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n), x_{n+1}) = f_{n+1}(x_1, \cdots, x_{n+1})$.

Prove that $f_n(x_1, \cdots, x_n) = (x_1 + \cdots + x_n)/n$.

Solutions. Part I

1. Let P be a polynomial and suppose that $x^n - 1/x^n = P(x - 1/x)$. It is clear that the degree of P must be n and the leading coefficient must be $+1$ to produce the term x^n in the right member of the identity. But then $-1/x^n = (-1)^n/x^n$ and hence n is odd. Since

$$x^{n+2} - x^{-n-2} = (x - x^{-1})^2(x^n - x^{-n}) - (x^{n-2} - x^{-n+2}) + 2(x^n - x^{-n}),$$

it follows that $x^{n+2} - x^{-n-2}$ can be represented as a polynomial in $x - x^{-1}$ provided $x^n - x^{-n}$ and $x^{n-2} - x^{-n+2}$ have such a representation. The representations for $x - x^{-1}$ and $x^3 - x^{-3}$ are immediate, and the induction proves the result true for all odd n.

2. Let z_1, z_2, z_3 be any three distinct points in the plane. These points form an equilateral triangle if and only if $z_3 - z_1 = (z_2 - z_1)\exp(\pm\frac{1}{3}\pi)$, where the same notation is used for a point and its representation as a complex number. Hence, $z_3 = -[-1 + \exp(\pm\frac{1}{3}\pi)]z_1 - [-\exp(\pm\frac{1}{3}\pi)]z_2$. The bracketed quantities are clearly the two nonreal cube roots of 1. Conversely, if $z_3 = -\omega z_1 - \omega^2 z_2$, where ω is a nonreal cube root of 1, then ω and ω^2 can be identified with the brackets for one choice of sign and the result follows.

3. Since $f(z) + zf(1 - z) = 1 + z$, it follows by substitution of $1 - z$ for z that $f(z) + z(2 - z - (1 - z)f(z)) = 1 + z$. Hence $f(z)(1 - z + z^2) = 1 - z + z^2$ for all z. Hence $f(z) \equiv 1$ except possibly for $z = w_1, w_2$, where these are the zeros of $1 - z + z^2$. Let α be an arbitrary complex number and $f(w_1) = \alpha$. Then since $w_2 = 1 - w_1$, we must have $f(w_2) = 1 + w_2 - w_2\alpha$. Thus $f(w_1)$ and $f(w_2)$ are arbitrary except for the relation indicated.

4. If $|xy - f(x) - g(y)| < \frac{1}{4}$ for all x and y in the unit interval, then, in particular, $|f(0) + g(1)| < \frac{1}{4}$, $|f(0) + g(0)| < \frac{1}{4}$, and $|f(1) + g(0)| < \frac{1}{4}$. But then by the triangle inequality

$$\begin{aligned} |1 - f(1) - g(1)| &\geq 1 - |f(1) + g(1)| \\ &\geq 1 - |f(1) + g(0)| - |-g(0) - f(0)| - |g(1) + f(0)| > \tfrac{1}{4}. \end{aligned}$$

5. Let a pursuer and the prey be at the points $(h, 0)$ and $(0, 0)$, respectively, of the usual Cartesian plane at time $t = 0$. If the prey moves along the positive x-axis at a constant speed v, then the condition that the pursuer moves directly

toward the prey at all times is given by $x-tv=(dx/dy)y$, where y has been taken as the independent variable. If the pursuer moves with speed $s>v$, then

$$tv = \frac{v}{s}\int_y^h \sqrt{\{1 + (dx/dy)^2\}}\,dy.$$

Substituting for t in the first equation and then differentiating to eliminate the integral, the relation becomes

$$y\frac{d^2x}{dy^2} = \frac{v}{s}\sqrt{\{1 + (dx/dy)^2\}}.$$

A first integral with the proper constant of integration is

$$\left(\frac{y}{h}\right)^{v/s} = \frac{dx}{dy} + \sqrt{\{1 + (dx/dy)^2\}}.$$

This last relation reduces to

$$2\frac{dx}{dy} = \left(\frac{y}{h}\right)^{v/s} - \left(\frac{y}{h}\right)^{-v/s}$$

and integrates to yield

$$2x - \frac{2(v/s)h}{1-(v^2/s^2)} = \frac{h(y/h)^{(v/s)+1}}{(v/s)+1} - \frac{h(y/h)^{-(v/s)+1}}{1-(v/s)}$$

with the proper initial conditions. Clearly the pursuit ends at

$$x = \frac{vsh}{s^2 - v^2}$$

at time $t=(sh)/(s^2-v^2)$ and the pursuer has gone a distance $(s^2h)/(s^2-v^2)$. In the case of the hawk and sparrow, $x=200/3$ ft. and $t=200/(3v)$. In the case of the eagle, $h=50$ ft. and his speed s is to be determined. The condition of the problem requires

$$\frac{50s}{s^2 - v^2} = \frac{200}{v} \quad \text{or} \quad 4s^2 - 3sv - 4v^2 = 0.$$

Hence the eagle's speed is $(\frac{3}{8}+\frac{1}{8}\sqrt{73})v$.

6. Let a_{ij}^k denote the element in the ith row and jth column of the matrix A_k. Take

$$a_{ij}^k = 1,\ 1 < i \leqq n,\ 1 \leqq k \leqq m; \qquad a_{11}^k = a_k,\ 1 \leqq k \leqq m;$$

$$a_{21}^2 = 1,\ a_{12}^1 = m\sum_{j=1}^{m} a_j - m^{2-n}a_{m+1};$$

and all other elements not yet specified as zero. The matrices A_k have the desired property.

7. Let $S=\{x|f(x)=0\}$. If S were infinite it would have an accumulation point x_0, and it is readily verified that $f(x_0)=f'(x_0)=0$, a contradiction. Hence S is finite. Therefore there is a polynomial h such that if $x\in S$ then $f'(x)h(x)=-1$. It follows that there is a (relatively) open set $Q\supset S$ such that if $x\in Q$ then $f(x)h'(x)-f'(x)h(x)>0$. If $c>0$, let $g_c(x)=xf(x)+ch(x)$, $a\leqq x\leqq b$. Then

$$f(x)g_c'(x)-f'(x)g_c(x)=f^2(x)+c(f(x)h'(x)-f'(x)h(x)).$$

If $x\in Q$, this expression is positive. Since $fh'-f'h$ is a bounded function and $f^2(x)$ is bounded away from 0 on the complement of Q, $fg_c'-f'g_c$ is a positive function on the complement of Q for sufficiently small c.

Solutions. Part II

1. Each of the points of intersection interior to the first quadrant corresponds in a unique way to a choice of two points on the X-axis and of two points on the Y-axis, namely the endpoints of the two segments intersecting in the given point. Conversely each such choice of four points determines uniquely one intersection point interior to the first quadrant. Since there are ${}_mC_2$ ways of choosing two points on the X-axis, and ${}_nC_2$ ways of choosing two points on the Y-axis, there are ${}_mC_2\cdot{}_nC_2=\frac{1}{4}m(m-1)n(n-1)$ points of intersection interior to the first quadrant.

2. For a given positive number, c, let p be the least integer for which $1/(10p)<c$. Then we can construct a series whose sum is c beginning with $1/(10q)$ for every integer $q\geqq p$. Since these series all have different initial terms, they are all different. The construction of any series is as follows. Starting with $1/(10q)$, add terms in order from the given series, always omitting any term which would bring the partial sum to c or beyond c. The partial sums thus formed will be monotonically increasing and will all be less than c. The error will be less than the last term omitted from the original series at any stage, so that the error tends to zero, and the constructed series converges to c. (It is possible to construct the series so that no two of the infinitely many series constructed have any terms in common, but this is not required by the problem.)

3. An example can be constructed using the Peano space-filling curve. Consider the Cantor set, C, of those numbers x in $0\leqq x\leqq 1$ which can be written in the ternary scale using only 0's and 2's. This may require an infinite sequence of 2's. (*e.g.*, $\frac{1}{3}=0.0222\cdots$). For all x's in C, define $f(x)$ by $f(.a_1a_2a_3\cdots)=.b_1b_2b_3\cdots$, where $b_i=0$ when $a_i=0$, and $b_i=1$ when $a_i=2$. Moreover, $f(x)$ is understood to be in the binary scale. The complement of the Cantor set is composed of intervals. Complete the definition of $f(x)$ in these intervals by requiring it to be linear and continuous. It is readily verified that $f(x)$ as defined is continuous. To show that it takes on every value infinitely many times, let y_0 be any number in $0\leqq y\leqq 1$. Then only the odd digits in the ternary expansion of the corresponding x are determined, and there are infinitely many different possible

values of x, formed from all choices of the even digits.

4. A possible attack is to consider possible least elements in allowable sets of five numbers. Since the largest element in the third row of the matrix is 18, no set of five can have a minimum greater than 18, since one element must be chosen from each row and each column. However, 18 is not possible as minimum, for 24 would have to be used in the second row, and then neither 22 nor 23 could be used from the fourth row, requiring the use of a number less than 18. Neither 16 nor 17 is a possible minimum, for if either is used, 25 can not be used, so that a number less than 16 or 17 would be required in the third column. However, 15 is a possible minimum. If 15 is the minimum, 25 must be used from the third column. Then in the second column, since 25 excludes 17, 20 must be used. In the first column, since 15 excludes 24, 23 must be used. Finally in the fifth column, 18 is the only element whose row is not yet occupied. With 15 as minimum we obtain the unique set 23, 20, 25, 15, 18. The above construction proves that the minimum of this set, 15, is as large as possible.

5. The given lines are not parallel, and it is readily verified that they do not intersect. Hence there is a unique line which is perpendicular to both of the given lines and intersects them both. The segment of this line between its two points of intersection is a diameter of the required sphere. To find the two points of intersection with the common perpendicular, minimize the distance between a pair of points, one from each line, *i.e.*, minimize $D^2=(t+1-4s+12)^2+(2t+4+s-8)^2+(-3t+5-s-17)^2$. The minimum is found to occur for $t=-\frac{782}{251}$, $s=\frac{657}{251}$. The corresponding points on the two lines are $(-\frac{521}{251}, -\frac{560}{251}, \frac{3601}{251})$ and $(-\frac{384}{251}, \frac{1351}{251}, \frac{4924}{251})$ respectively. The midpoint of the segment between these points is $(-\frac{915}{502}, \frac{791}{502}, \frac{8525}{502})$ and this is the center of the required sphere. The radius of the sphere is the distance from the center to either of the two points of intersection, and is $\frac{147}{502}\sqrt{251}$. The equation of the sphere can be written $(502x+915)^2+(502y-791)^2+(502z-8525)^2=5{,}423{,}859$.

6. Since x and y are both positive, $1/x$ and $1/y$ are both less than 1, so that both x and y are greater than 1. As a result, no two multiples of x have the same integral part, and no two multiples of y have the same integral part. Thus no integer appears more than one time in either of the two sequences. Suppose that an integer N appeared in both sequences. Then we could find integers p and q such that $N<px<N+1$ and $N<qy<N+1$. (No equality is possible since x and y are irrational.) Solving these inequalities for $1/x$ and $1/y$,we find

$$\frac{p}{N+1}<\frac{1}{x}<\frac{p}{N}, \qquad \frac{q}{N+1}<\frac{1}{y}<\frac{q}{N}.$$

Adding we find

$$\frac{p+q}{N+1}<1<\frac{p+q}{N} \quad \text{or} \quad N<p+q<N+1,$$

which is impossible since N, p, and q are all integers. Finally suppose that an

integer M is missing from both sequences. Then we can find integers p and q such that $px<M$, $(p+1)x>M+1$, $qy<M$, $(q+1)y>M+1$. Solving for $1/x$ and $1/y$ as before, we are led to $M-1<p+q<M$, which is again impossible. Therefore every integer is present in one sequence or the other, and each integer occurs exactly once.

7. The result will be proved by induction. From (2), $f_1(0)=f_1(-0)=-f_1(0)$, so that $f_1(0)=0$. Then from (1), $f_1(x)=f_1(0+x)=f_1(0)+x=0+x=x$. Hence the theorem is true for $n=1$. Now assume the theorem true through n. We first note that for any A, $f_{n+1}(0, \cdots, 0, A)=A/(n+1)$. To see that, start with

$$\begin{aligned} f_{n+1}(0, \cdots, -B, B) &= f_{n+1}(0, \cdots, B, -B) && \text{(by symmetry)} \\ &= -f_{n+1}(0, \cdots, -B, B) && \text{(by (2)).} \end{aligned}$$

Hence $f_{n+1}(0, \cdots, -B, B)=0$. But, by the induction hypothesis, $f_n(0, \cdots, 0, -B)=-B/n$, so that by (3),

$$0=f_{n+1}\left(-\frac{B}{n}, \cdots, -\frac{B}{n}, B\right)=f_{n+1}\left(0, \cdots, 0, \frac{n+1}{n}B\right)-\frac{B}{n}$$

by (1). Setting $B=(nA)/(n+1)$, we get $f_{n+1}(0, \cdots, 0, A)=A/(n+1)$. Now start with (3), writing $\bar{x}_n=(x_1+\cdots+x_n)/n$:

$$\begin{aligned} f_{n+1}(x_1, \cdots, x_n, x_{n+1}) &= f_{n+1}(\bar{x}_n, \cdots, \bar{x}_n, x_{n+1}) && \text{(induction hypothesis)} \\ &= f_{n+1}(0, \cdots, 0, x_{n+1}-\bar{x}_n)+\bar{x}_n && \text{(by (1))} \\ &= \frac{x_{n+1}}{n+1}-\frac{\bar{x}_n}{n+1}+\bar{x}_n && \text{(by the lemma with } A=x_{n+1}-\bar{x}_n) \\ &= \frac{x_1+\cdots+x_{n+1}}{n+1}. \end{aligned}$$

Hence the theorem is true for $n+1$ if it is true for n and, by induction, it is true for all n.

A CONFERENCE ON MATHEMATICS CURRICULA IN INSTITUTES

A Conference on Mathematics Curricula in NSF Institutes for High School Teachers, sponsored and planned by the MAA Committee on Institutes and financed by the National Science Foundation, was held in Washington, D. C., on September 10 and 11, 1960. This report is presented by the MAA Committee on Institutes as representing the consensus of the Conference. The organizations represented and the mathematicians in attendance are listed at the end of the report.

Background, purposes, and present status of institutes. From a modest beginning of two NSF summer institutes in 1953, this activity has increased to 396 NSF summer institutes in 1960, 33 academic year institutes in 1960–61, and 204 in-service institutes in 1960–61. The vast majority (89%) of these institutes are for high school teachers, and mathematics teachers are included in 212 summer institutes, in 32 academic year institutes, and in 136 in-service institutes. Approximately 13,000 mathematics teachers will attend an institute during the summer of 1960 or the academic year 1960–61.

The ultimate purpose of an institute for high school mathematics teachers is to improve the mathematical education of the students in our schools. Institutes aim to accomplish this purpose by increasing the mathematical competence and background of the teachers. Thus, courses and procedures in institutes should be relevant to the courses which the institute participants will teach in their classrooms.

NSF institutes constitute one of the most effective means of improving the teaching of high school mathematics. They have proved their efficacy in helping high school teachers do a better job with traditional courses and in enabling them to teach more modern courses. They seem to be the best hope of implementing the curricula changes being recommended by the various national agencies. Thus, the quality and effectiveness of mathematics courses in institutes is a matter of importance to the entire mathematical community.

It seems likely that institutes will be a heavy obligation on the mathematics profession for some years to come. Hence, it is essential that qualified small colleges, as well as large institutions, be engaged in the conduct of mathematics institutes, particularly in-service and summer institutes. As institutions and staffs with less experience undertake to conduct institutes, it becomes increasingly desirable to provide assistance to newcomers in institute work, to facilitate the exchange of information, and to encourage a certain degree of standardization of courses. The MAA Committee on Institutes can be of considerable service in this connection, and its functions should be continued and expanded. During the past year, the Committee has distributed to directors a list of possible mathematics instructors for institutes, it has called the attention of departmental chairmen to the need for more proposals for institutes of certain types, and it has conducted five regional conferences for mathematics lecturers in institutes. It was suggested that the Committee work in coordination with CUPM to prepare somewhat detailed guidelines for courses suitable to institutes.

In-service and academic year institutes have their own peculiar problems. While this report refers primarily to summer institutes, much of what it says applies to the other types, particularly the statements about courses.

Types and structure of institutes. Institutes should be planned for specified levels of teaching such as junior high school, senior high school, etc., and for specified levels of mathematical background—below-average, average, above-

average—on the part of teachers. The five-level teaching classification proposed by CUPM in the December 1960 MONTHLY might be adopted; in which case the recommendations of this report would be primarily applicable to levels II and III. In general, an institute should be planned for one teaching level, but in special cases institutes including participants from more than one level may have advantages. In any case relative homogeneity within a given course is highly desirable.

Sequential institutes, that is institutes planned to continue participants for two or more summer or in-service programs, are becoming frequent. They serve a good purpose in offering sustained training, which in some cases culminates in a master's degree. Such institutes have opportunities for courses at increasing levels of sophistication and can prepare teachers for teaching advanced twelfth grade courses such as the one in the Advanced Placement Program. An alternative plan, the success of which depends upon some standardization of courses, is for different colleges to operate institutes of different types and to encourage participants to attend institutes of progressively "higher" type in successive summers.

Procedures and materials. Courses should be conducted in such a fashion as to reassure the participants and make them feel at ease. High school teachers are usually rusty and frequently sensitive about their weaknesses. Generally they are hard-working conscientious people who respond as well as they can to encouragement and assistance.

Regular conference periods in which the participants work with the direct help of the lecturer or of skilled and knowledgeable assistants should be a prominent feature of an institute program. Also, ample opportunity for informal contact between the lecturers and the participants should be provided.

It is desirable that a text, a small group of texts, or prepared mimeographed notes be used in each course, and that these materials be available at the time of study in the course.

Staff. The quality of an institute's instructional staff is the most important factor in the success of the institute. This quality is measured by the instructor's knowledge of mathematics, his enthusiasm for mathematics at the level of the participants, his sensitivity to their reactions, and his ability to communicate effectively with them. He should be fully aware of the mathematics taught in high school and the relation his course bears to that high school mathematics.

Securing adequate staff is the most difficult and most important problem for institute directors. Continuing efforts should be made to seek out and interest appropriate people to teach institute courses.

Courses. Institute courses should be planned with careful regard to the background, ability, and needs of the participants. They should be designed to deepen understanding of the mathematics which the participants will teach. They should include both a study of the ideas underlying high school courses

and an indication of the nature of the mathematics being taught after high school.

Since institute participants generally will be teaching algebra or geometry or both, courses in these subjects should be included in the *first* institute experience of most participants.

Algebra. The first algebra course is perhaps the most basic of all the institute courses, since algebra constitutes such a large part of high school mathematics. Most teachers have no clear concept of algebraic structure or real understanding of the number systems of elementary algebra. A first institute course in algebra should aim to alleviate these deficiencies.

A few general principles concerning algebra courses were agreed upon. The content of the course should be chosen for its relevance to high school algebra. In the first algebra course number systems must be dealt with in some manner. The number systems furnish the motivation for the study of abstract systems, and the treatment of these last should be such as to throw light upon the ordinary number systems by putting the latter in the proper perspective. The course should not be too narrow, but, at the same time, a few concepts should be carefully developed. The language and symbolism of sets should be introduced and used. Algebra is appropriate material for teaching postulational methods and simple principles of logic. The course should contain some material which is new to the participant. Current experimental high school texts with the accompanying manuals and texts for teachers can be helpful in devising appropriate courses.

Geometry. The general outline of geometry courses is not as clear as in the case of algebra, nor are textbooks as readily available. Considerable reliance upon recently developed programs for high school geometry is perhaps indicated. Conceivably an institute course in geometry could be built upon one of the recently proposed high school courses with extensive insertions at the proper places. If a course is modeled upon some particular one of these programs, care should be taken to make clear that the course represents only one of several logically correct ways to develop Euclidean geometry.

There was much discussion of the intuitive approach *vs* deductive reasoning, particularly in the case of a course for junior high school teachers. The latter group perhaps needs a special course in which the intuitive aspect is emphasized. In the first course in geometry the language of sets should be used naturally, and, unless certain matters of logic are treated elsewhere, they must receive attention in this course.

In subsequent courses, various possibilities were suggested. There is much to be said for the study of one or more geometries other than Euclidean. There is a large variety of topics from higher geometries that can be fitted into courses. The big ideas of elementary geometry—measure, congruence, similarity, convexity, etc.—can be considered from an advanced point of view. A course stress-

ing applications and designed to enhance the participants' intuition is another possibility. A careful treatment of analytic geometry in which the postulates are those characterizing the real numbers and the geometric objects are defined entities is also worthy of consideration.

Analysis. A distinction should be made between a first course in analysis for the usual high school mathematics teacher and special courses serving to prepare for teaching the advanced placement course in calculus and analytics.

A first course in analysis for the generality of participants should present the essential concepts of calculus with more emphasis upon ideas than devices. A judicious selection of a small number of basic theorems should be made for careful proof. The course should give teachers some acquaintance with the content of mathematics courses beyond high school and the high school skills required for mastering the same. Moreover, it should illuminate parts of the high school mathematics program by increasing the participants' knowledge of the structure of the real number system, the function concept and properties of particular functions such as the logarithmic and exponential functions, limits of sequences and sums of infinite series, and areas bounded by curves.

Preparation for teaching the Advanced Placement Program course requires distinctly more than the introductory course discussed in the preceding paragraph. Such preparation is appropriately obtained from an academic year institute or from a sequence of appropriately graded summer institute courses.

A special type of institute for training teachers to teach the Advanced Placement course might include high school teachers who already have fair competence in elementary calculus, junior college teachers, and elementary calculus teachers in four-year colleges.

Probability and Statistics. It was agreed that courses in probability and statistics should be included in institutes *only after* algebra and geometry—and perhaps also analysis—are taken care of. There was also strongly held opinion on the part of some that further work in such branches of pure mathematics as number theory would be of greater value to participants. In any case, it was held that emphasis should be on probability rather than statistics, but there was objection to excluding statistics altogether. A number of texts are now available for a first course in probability. Adequate preparation to teach a course in probability in high school would require more than a single course.

Foundations and Logic. There was some division of opinion regarding the best way to handle questions of foundations and logic. Courses in formal logic were not considered appropriate, but it was recognized that some attention must be paid to elementary principles of logic. The suggestions were essentially of two kinds: to treat these principles as they arise naturally in mathematics courses, or to devote a few days in some one course to discussing them. The treatment of certain topics in algebra and geometry courses may be such as to illustrate adequately the basic logical principles.

In academic year institutes and sequential summer institutes, courses in other topics such as number theory, linear algebra, elementary topology, computation, etc., may well be appropriate. Cardinal principles should be to include the essential courses first and to tailor all courses to the level of the participants.

Academic activities other than courses. Seminars on high school curricula have proved valuable in acquainting the participants with various current curriculum enterprises, and in relating the material being taught in the institute courses to high school mathematics courses.

If a demonstration class is part of an institute program, arrangements should be made to supplement meetings of the class with introductory or follow-up discussions between the teacher of the class and the participant observers.

Visiting lecturers for short periods can add interest to an institute, but can easily be overdone. The participants need plenty of time to study, and features outside of courses should not be allowed to interfere. In any case, visiting lecturers should be carefully chosen and requested to talk about something in the range of comprehension of the participants, preferably something which fits in with the overall plan of the institute.

Academic credit and degrees. Most teachers attending institutes want graduate credit for the courses taken. This desire is due to certification requirements and conditions governing salary increases. Many institutions are handling this matter by awarding graduate credit for institute courses which can count only toward a special type of master's degree, such as a Master of Arts in the Teaching of Mathematics. Such degrees, which allow most of the required courses to be in mathematics with only a minimum requirement in education, have been inaugurated recently in a large number of institutions.

Appendix. Organizations represented at the Conference: MAA Committee on Institutes, MAA Committee on the Undergraduate Program, School Mathematics Study Group, AAAS Cooperative Committee on the Teaching of Science and Mathematics, National Council of Teachers of Mathematics, National Association of State Directors of Teacher Education and Certification.

Persons attending the Conference: C. B. Allendoerfer, R. D. Anderson, H. M. Bacon, E. G. Begle, S. J. Bezuszka, W. E. Briggs, C. F. Brumfiel, E. A. Cameron, J. W. Cell, Roy Dubisch, W. L. Duren, W. T. Guy, Jr., S. P. Hughart, P. S. Jones, Joseph Landin, N. H. McCoy, W. H. L. Meyer, E. E. Moise, H. T. Muhly, W. W. Osborn, C. R. Phelps, G. B. Price, H. W. Syer, G. B. Thomas, Marie S. Wilcox, N. B. Winters, R. J. Wisner.

Members of the MAA Committee on Institutes: E. A. Cameron, Chairman; E. G. Begle, W. T. Guy, Jr., Kenneth May, W. H. L. Meyer.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

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INFINITE PRODUCTS FOR πe AND π/e

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Let S_n be a closed solid n-dimensional sphere and let C_n be the solid n-dimensional cylinder of largest volume that can be inscribed into S_n; $V(S_n)$ and $V(C_n)$ will denote the volumes of the respective solids. It is shown first that

$$\lim_{n\to\infty} V(C_n)/V(S_n) = (2/\pi e)^{1/2}, \tag{1}$$

and next that

$$\pi/2e = \prod_{n=1}^{\infty} (1 + 2/n)^{(-1)^{n+1}n} \tag{2}$$

and

$$6/\pi e = \prod_{n=2}^{\infty} (1 + 2/n)^{(-1)^n n}. \tag{3}$$

The derivation of (2) and (3) from (1) parallels closely Wallis' derivation of the infinite product for π.

Let E^n be the n-dimensional Euclidean space equipped with a Cartesian coordinate system. Use will be made of the vector notation: $x = (x_1, x_2, \cdots, x_n)$ will denote both the point and the corresponding vector (from the origin); $x \cdot y$ is the scalar product, and so on. A closed solid sphere S_n is any set of points x, such that $(x-a)\cdot(x-a) \leqq r^2$; here r is the radius and the point a is the center of S_n. A hyperplane H is any set of points x, such that $a \cdot x = b$ and $a \cdot a = 1$. The vector a is the direction of H. Two hyperplanes H_1 and H_2 are parallel if their equations are $a \cdot x = b_1$, and $a \cdot x = b_2$, $b_1 \neq b_2$. The distance d between H_1 and H_2 is then $d = |b_1 - b_2|$.

Let H_1 and H_2 be two parallel hyperplanes in the direction a and let K be any closed set contained in H_1. Let $[xy]$ denote the closed straight segment with the end points x and y. A closed solid cylinder C is any set of the following form: C consists of all points z such that z lies on a segment $[xy]$ where x is in K, y is in H_2, and the vector $x-y$ is in the direction a. The hyperplanes H_1 and H_2 will be called the bases of the cylinder C, the distance d between H_1 and H_2 is the height of C, and the set K is the directrix of C. A cylinder C is called spherical if its directrix is a sphere (of some dimension). (The above definition reduces for $n=3$ to the usual definition of a right cylinder.)

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Let C be the cylinder of largest volume, inscribable into a sphere S_n of unit radius. Strictly speaking, one should prove first that this largest cylinder exists. Such a proof could be easily deduced from Blaschke's selection principle [1], [2], but it will not be given here. Let H_1 and H_2 be the bases of C and let K_1 and K_2 be their intersections with S_n. Let $d=2h$ be the distance between H_1 and H_2. Now C must lie within the spherical cylinder C_1 with the directrix K_1 and height d, and C must also lie within the spherical cylinder C_2 with the directrix K_2 and height d. The bases of C_1 and C_2 are H_1 and H_2 and their directrices are the $(n-1)$-dimensional spheres K_1 and K_2 of radii r_1 and r_2 respectively.

If follows that C lies in the intersection, C_3, of C_1 and C_2. Since C is maximal and C_3 is a cylinder inside S_n, it follows that $C=C_3$, that $r_1=r_2=r$, and that the center o of S_n lies half-way between H_1 and H_2.

It is shown in any text on analysis, *e.g.*, [3], that the volume V of a closed solid n-dimensional sphere of radius r is

$$V = r^n\pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1). \tag{4}$$

The volume of a spherical cylinder is the product of its height and the volume of its spherical base. Hence for the sphere S_n of unit radius and for its inscribed maximal cylinder C_n one gets

$$V(S_n) = \pi^{n/2}/\Gamma(\tfrac{1}{2}n + 1), \tag{5}$$

$$V(C_n) = 2h\pi^{(n-1)/2}r^{n-1}/\Gamma(\tfrac{1}{2}(n-1) + 1). \tag{6}$$

The quantities r and h are not independent and in fact

$$r^2 + h^2 = 1, \tag{7}$$

since the radius of S_n is 1.

To make sure that $V(C_n)=$ maximum, one has to maximize in (6) the product hr^{n-1} under the side-condition (7). This is easily accomplished and the maximum of hr^{n-1} is found to be

$$n^{-1/2}(1 - 1/n)^{(n-1)/2}. \tag{8}$$

Now let

$$\rho_n = V(C_n)/V(S_n); \tag{9}$$

by (5), (6), and (8)

$$\rho_n = 2(\pi n)^{-1/2}(1 - 1/n)^{(n-1)/2}\Gamma(\tfrac{1}{2}n + 1)/\Gamma(\tfrac{1}{2}n + \tfrac{1}{2}). \tag{10}$$

Observing that

$$\lim_{x\to\infty} \Gamma(x + \alpha)/x^\alpha\Gamma(x) = 1,$$

([4], p. 254) and recalling the definition of e, one gets from (10) by passing to the limit

$$\rho = \lim_{n\to\infty} \rho_n = (2/\pi e)^{1/2}, \tag{11}$$

which proves (1). To prove (2) and (3) one calculates first $\sigma_n = \rho_{n+2}/\rho_n$. This expression is used in order to apply the difference equation of the Γ-function. One has by (10)

$$\sigma_n = [n/(n+2)]^{n/2}[(n+1)/(n-1)]^{(n-1)/2}. \tag{12}$$

Simple calculation shows that

$$\rho_2 = 2/\pi, \qquad \rho_3 = 3^{-1/2}. \tag{13}$$

Now one has

$$\rho = \rho_2 \prod_{n=1}^{\infty} \sigma_{2n}, \qquad \rho = \rho_3 \prod_{n=1}^{\infty} \sigma_{2n+1}. \tag{14}$$

Hence by (11), (12) and (13)

$$(2/\pi e)^{1/2} = 2/\pi \prod_{n=1}^{\infty} [n/(n+1)]^n[(2n+1)/(2n-1)]^{(2n-1)/2}, \tag{15}$$

$$(2/\pi e)^{1/2} = 3^{-1/2} \prod_{n=1}^{\infty} [(2n+1)/(2n+3)]^{(2n+1)/2}[(n+1)/n]^n. \tag{16}$$

Squaring and simplifying (15) and (16) yields (2) and (3). Convergence proofs for the products (2) and (3) are easily deduced from the standard criteria for infinite products, [4]. New infinite products for e^2 and π^2 are obtained on multiplying the expressions in (2) and (3). The first four partial products in (2) are: .75, .686 $\cdots$, .657 $\cdots$, .639 $\cdots$; as might be expected, convergence to $\pi/2e = .57786\cdots$ is rather slow.

Two conjectures will be made in conclusion:

(A) *π/e is irrational and the irrationality can be proved by starting with* (2).

(B) *Let S be a closed convex bounded n-dimensional set with nonempty interior, then there is in S a convex n-dimensional cylinder C, such that $V(C)/V(S) \geqq k > 0$ and the constant k does not depend on the dimension n. By considering the case of a cone it is easy to show that $k \leqq 1/e$.*

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A POLYNOMIAL IN EACH VARIABLE SEPARATELY IS A POLYNOMIAL

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The result in this note is probably not new, although I have been unable to find it in the literature. Perhaps the most surprising aspect of the problem is that the proof, while elementary, is not trivial.

THEOREM. *Let $f: R\times R\to R$ have the property that for each fixed value of x (respectively, y), $f(x, y)$ assumes the values of a polynomial $P_x(y)$ (respectively, of a polynomial $P_y(x)$). Then f is a polynomial on $R\times R$.*

Proof. It follows from the hypothesis that

$$f(x, y) = \sum_{n=0}^{\infty} a_n(y)x^n, \tag{1}$$

where, for each y, $a_n(y)=0$ for all but finitely many n. Since R is not a countable union of finite sets, there exists an integer N such that the set $F=\{y: a_n(y)=0$ for all $n>N\}$ is infinite. Denoting by f_1 the restriction of f to $R\times F$, we have

$$f_1(x, y) = \sum_{n=0}^{N} a_n(y)x^n, \qquad (x, y)\in R\times F. \tag{2}$$

Choosing $N+1$ distinct values $x_0, x_1, \cdots, x_N$, substituting them for x in (2), and solving the resulting system of equations, we obtain

$$a_n(y) = \sum_{j=0}^{N} c_{jn}f_1(x_j, y) \qquad (y\in F, n = 0, \cdots, N), \tag{3}$$

where the c_{jn} are real constants. Thus the function g, defined on $R\times R$ by

$$g(x, y) = \sum_{n=0}^{N}\sum_{j=0}^{N} c_{jn}f(x_j, y)x^n, \tag{4}$$

is a polynomial. Moreover, (2) and (3) show that, for each $x'\in R$, the polynomial $f(x', y)-g(x', y)$ has a zero at each point of F, hence is equal to zero for all y.

A NOTE ON FACTORIZABLE GROUPS

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In the classification of groups the question whether or not a group can be written as a product of two of its subgroups often arises. In this note we will prove with the aid of a lemma of complementation theory of groups that a certain torsion group is completely factorizable.

DEFINITION. *A group G is factorizable if $G=AB$ for subgroups A and B in G. If in addition $A\cap B=1$, the identity of G, G is completely factorizable.*

DEFINITION. *Let π be a given set of primes. The group H is a π-group if the prime divisors of the orders of the elements of H belong to π.*

DEFINITION. *A normal subgroup A of a group G is complemented in G if there exists a subgroup F of G such that $G=AF$ and $A\cap F=1$.*

In the following lemma π is a given set of primes, and $(n, \pi)=1$ means $(n, p)=1$ for all p in π.

LEMMA. *Suppose A is a normal, solvable, torsion, π-subgroup of finite index n in the group G. If $(n, \pi)=1$, then A is complemented in G.*

Proof. We will prove this lemma by induction on the length k of the solvable series of A. If $k=1$, then A is an abelian group, and the proof for finite groups given in [1], pages 201–202, is applicable. Suppose the lemma is true for groups satisfying the conditions of the lemma and for normal subgroups whose solvable series are of length k, and suppose the length of the solvable series of A is $k+1$. The commutator subgroup A' of A is normal in G since it is characteristic in A and A is normal in G. The group A/A' is an abelian, torsion, π' subgroup of index n in G/A', where π' is a subset of π. Since the lemma has been proved for $k=1$, there exists $\overline{H}\subset G/A'$ such that $G/A'=\overline{H}(A/A')$ and $\overline{H}\cap(A/A')=1$. Let H be the complete inverse image of $\overline{H}$. By the isomorphism theorem for groups, HA/A is isomorphic to $H/H\cap A$. Thus, the index of $H\cap A$ in H is n which is prime to π by hypothesis. Since $\overline{H}\cap(A/A')=1$, $H\cap A=A'$. The length of the solvable series of A' is k. By the induction hypothesis there exists a subgroup $F\subset H$ such that $H=A'F$ and $A'\cap F=1$. Now $G=HA=FA'A=FA$. Since $(n, \pi)=1$ and the order of F is n, $F\cap A=1$.

Suppose G is the group of permutations on the symbols 1, 2, 3, and 4. The commutator subgroup G' of G is the alternating group. The group G contains 3 Sylow 2-subgroups of order 8 which we will designate by P_1, P_2, and P_3. The intersection $P_*=P_1\cap P_2\cap P_3$ is the group K whose elements are (12) (34), (13) (24), (14) (23), and the identity. Now $P_1\cap G'=K=P_*\cap G'$. Hence, $P_1\cap G'\subset P_*\cap G'$. Thus the group G satisfies the hypotheses of the following theorem.

THEOREM. *Suppose the torsion group G contains a solvable Sylow π-subgroup P of finite index in G. If $P_*\cap G'\supset P\cap G'$, where P_* is the intersection of the complete set of conjugates of P, then G is completely factorizable.*

Proof. Under the natural homomorphism of G onto G/G' the image of P, $\overline{P}$, is a Sylow η-subgroup of G/G' where η is a subset of π. Since G/G' is torsion and abelian and since $\overline{P}$ is a Sylow η-subgroup of G/G', G/G' can be expressed as the direct product $\overline{P}\otimes\overline{Q}$ where the orders of the elements of $\overline{Q}$ are divisible by primes which are not in η. Let H be the complete inverse image of $\overline{P}$. The group H is generated by P and G' since H is generated by a set of representatives of the cosets of $\overline{P}$ and G' and the set of representatives can be taken in P. Since G' is normal in G, G' is normal in H, and $H=PG'$. Let Q be the complete inverse

image of $\overline{Q}$. Since Q is normal in G and since P has a finite number of conjugates, by ([1], p. 161 and 163), $P\cap Q$ is a Sylow λ-subgroup of Q where λ is a subset of π.

Now $P\cap Q\supset P\cap G'$. Suppose $P\cap G'\not\supset P\cap Q$. Then there is an element x in $P\cap Q$ which is not in $P\cap G'$, and therefore, not in G'. The image of x under the natural homomorphism is in $\overline{Q}$, and it is not the identity since x is not in G'. Since x is in $P\cap Q$, the order of x is divisible by primes in π. Hence the order of the image of x is divisible by primes in η. But $\overline{Q}$ contains no element whose order is divisible by primes in η. Hence $P\cap Q=P\cap G'$, and since $P_*\cap G'\supset P\cap G'$, $P_*\cap G'\supset P\cap Q$. Certainly $P\cap Q\supset P_*\cap G'$. Therefore $P\cap Q=P_*\cap G'$.

Since P is of finite index in G and since it has a finite number of conjugates, P_* is of finite index in G. Now $[Q: Q\cap P_*]$ is finite since it is equal to $[P_*Q: P_*]$. Since $[Q: Q\cap P_*]=[Q: Q\cap P][Q\cap P: Q\cap P_*]$ and since $[Q: Q\cap P_*]$ is finite, $[Q: Q\cap P]$ is finite. Now P_* is normal in G since the intersection of a complete set of conjugate subgroups in a group is normal in the group. Since $P\cap Q=P_*\cap G'$, $P\cap Q$ is normal in G, and hence in Q. Since $P\cap Q$ is a Sylow λ-subgroup of Q, $([Q: Q\cap P], \lambda)=1$. By the above lemma there exists a subgroup $F\subset Q$ such that $Q=(P\cap Q)F$ and $(P\cap Q)\cap F=1$. Now $G=HQ=PG'Q=PQ=P(P\cap Q)F=PF$. Since $F\subset Q$, $1=(P\cap Q)\cap F=P\cap F$.

Reference

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A DECOMPOSITION OF CONTINUITY IN TOPOLOGICAL SPACES

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0. Introduction. In the literature there are many "weakened" forms of continuity, *e.g.*, upper and lower semicontinuity, continuity almost everywhere, approximate continuity, sequential continuity and no doubt many others. In this note we (1) introduce two new weakened forms of continuity, (2) show they are independent, and (3) show that taken together, they characterize continuity in the most general cases. One of the weakened forms is characterized in terms of inverses of open sets. In addition, examples are given to illustrate the various connections between the two weakened forms of continuity, sequential continuity, and continuity.

I. Let $f: X\to X^*$ be a single valued function (not necessarily continuous), X and X^* being topological spaces.

DEFINITION 1. *$f: X\to X^*$ will be termed weakly continuous at $x\in X$ if and only if for $f(x)\in 0^*$ open in X^*, there is an open set 0 in X such that $x\in 0$ and $f(0)\subset c^*0^*$, c and c^* denoting the closure operators in X and X^* respectively. $f: X\to X^*$ will be termed weakly continuous (denoted henceforth as w.c.) if and only if $f: X\to X^*$ is weakly continuous at each of the points in X.*

* Now at The Ohio State University.

THEOREM 1. *Let $f: X \to X^*$ be single valued, X and X^* being topological spaces. Then $f: X \to X^*$ is w.c. if and only if for 0^* open in X^*, $f^{-1}(0^*) \subset \operatorname{Int} f^{-1}c^*0^*$, Int denoting the interior operator.*

Proof. Sufficiency. Let $x \in X$ and $f(x) \in 0^*$. Then $x \in f^{-1}(0^*) \subset \operatorname{Int} f^{-1}c^*0^*$. Let $0 = \operatorname{Int} f^{-1}c^*0^*$. $f(0) = f \operatorname{Int} f^{-1}c^*0^* \subset ff^{-1}c^*0^* \subset c^*0^*$.

Necessity. Let $x \in f^{-1}0^*$. Then there exists an open set 0 such that $x \in 0$ and $f(0) \subset c^*0^*$. Hence $x \in 0 \subset f^{-1}c^*0^*$ and $x \in \operatorname{Int} f^{-1}c^*0^*$.

THEOREM 2. *Let $f: X \to X^*$ be single valued and X^* regular. Then $f: X \to X^*$ is w.c. if and only if $f: X \to X^*$ is continuous.*

Proof. The sufficiency is clear. Necessity. Let $x \in X$ and $f(x) \in 0^*$. Then $f(x) \in 0^* \subset c^*0^* \subset 0^*$. There exists an open set 0 such that $x \in 0$ and $f(0) \subset c^*0^* \subset 0^*$. Thus $f: X \to X^*$ is continuous.

Example 1. w.c. does not imply sequential continuity. Let J denote the unit interval with the usual topology. Let S denote the unit interval with the following topology: a set A in S is open if and only if either $A = \phi$ or $\mathcal{C}A$ is countable, $\mathcal{C}$ denoting the complement operator. Let $I: J \to S$ be the identity mapping. The reader will easily verify that $I: J \to S$ is w.c. but not sequentially continuous.

Example 2. Sequential continuity does not imply w.c. Let $I: S \to J$ be the identity mapping, S and J being the spaces in Example 1. The reader will easily verify that $I: S \to J$ is sequentially continuous but not w.c.

Example 3. w.c. plus sequential continuity does not imply continuity. Let S be the space in Example 1 and $S^* \equiv \{a^*, b^*\}$ where the open sets are ϕ, $\{a^*\}$, and S^*. Let $f: S \to S^*$ as follows:

$$f(x) \equiv \begin{cases} a^* & \text{if } x \text{ is rational} \\ b^* & \text{if } x \text{ is irrational.} \end{cases}$$

Then $f: S \to S^*$ is not continuous since $f^{-1}(a^*)$ is not open in S, but is sequentially continuous as well as w.c. We leave the details to the reader.

II. We now introduce a complementary form of weak continuity.

DEFINITION 2. *$f: X \to X^*$ will be termed w*.c. if and only if when 0^* is open in X^*, then f^{-1} fr 0^* is closed in X, where* fr *denotes the frontier operator.*

Example 4. w.c. does not imply w*.c. Let $S \equiv \{a, b\}$ with open sets ϕ, $\{a\}$, and S and $S^* \equiv \{a^*, b^*\}$ with open sets ϕ, $\{b^*\}$, and S^* and let $f(x) \equiv x^*$ on S. Then $f: S \to S^*$ is clearly w.c. but not w*.c. since fr $b^* = c^*b^* \cap c^*a^* = a^*$. But $f^{-1}a^* = a$ and $\{a\}$ is not closed in X.

Example 5. w*.c. does not imply w.c. Let $S \equiv \{a, b\}$ with open sets ϕ and S and $S^* \equiv \{a^*, b^*\}$ with ϕ, $\{a^*\}$, $\{b^*\}$, and S^* as the open sets. Let $f(x) \equiv x^*$. Then $f: S \to S^*$ is not w.c. for $f(a) = a^* \in \{a^*\} = c^*a^*$. But the only open set con-

taining a is S and $f(S) \not\subset c^*a^*$. $f: S \to S^*$ is w*.c. for fr $a^* = a^* \cap b^* = \phi$ and $f^{-1}(\phi)$ is closed.

Example 6. Sequential continuity plus w*.c. does not impy w.c. Let S be the space in Example 1 and $S^* \equiv \{a^*, b^*\}$ with ϕ, $\{a^*\}$, $\{b^*\}$, and S^* the open sets. Define

$$f(x) \equiv \begin{cases} a^* & \text{if } x \text{ is rational} \\ b^* & \text{if } x \text{ is irrational.} \end{cases}$$

$f: S \to S^*$ is clearly sequentially continuous. Since fr $a^* = \phi = \text{fr } b^*$, f^{-1} fr 0^* is closed for all 0^* open in S^* and thus $f: S \to S^*$ is w*.c. But $f: S \to S^*$ is not w.c. since $f(\frac{1}{2}) = a^*$ and a^* is open in S^*. $c^*a^* = a^*$ and there is no open set 0 containing $\frac{1}{2}$ such that $f(0) = a^*$.

Example 7. Sequential continuity plus w.c. does not imply w*.c. See Example 3.

III. THEOREM 3. *Let $f: X \to X^*$ be single valued, X and X^* being topological spaces. Then $f: X \to X^*$ is continuous if and only if it is both* w.c. *and* w*.c.

Proof. The necessity is clear. Sufficiency. Let $f(x) \in 0^*$. Since $f: X \to X^*$ is w.c. there exists an open set 0 containing x and $f(0) \subset c^*0^*$. Now fr $0^* = c^*0^* - 0^*$ and thus $f(x) \notin \text{fr } 0^*$. Hence $x \notin f^{-1}$ fr 0^* and therefore $x \in 0 - f^{-1}$ fr 0^*, an open set since the transformation is w*.c. The proof will be complete when we show $f\{0 - f^{-1} \text{ fr } 0^*\} \subset 0^*$. To this end let $y \in 0 - f^{-1}$ fr 0^*. Then $y \in 0$ and hence $f(y) \in c^*0^*$. But $y \notin f^{-1}$ fr 0^* and thus $f(y) \notin \text{fr } 0^* = c^*0^* - 0^*$ which implies that $f(y) \in 0^*$.

CLASSROOM NOTES

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LINEAR PROGRAMMING AND APPROXIMATION PROBLEMS

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1. Introduction. In this note we attempt to propagandize in favor of the teaching of linear programming techniques in undergraduate numerical analysis courses. Although linear programming has been the instrument of a near-incredible tidal wave of large-scale industrial problem solving, and is admitted, even by its detractors, as a tool of undoubted merit in many applications,

notably in the areas of resource allocation and transportation, it is mentioned only fleetingly in most undergraduate curricula. The reasons for this are fairly transparent; not only is the subject new (and hence the victim of the customary lag between professional utility and curricular acceptance), but it has the added handicap that practically all except trivial examples are not easily susceptible to pencil-and-paper solution. Linear programming is the child of the young age of the high-speed, large-memory electronic digital computer; the algorithms for solution of linear programming problems have been designed with computer utility in mind. The simplex method and its invariants do not lend themselves to efficient pencil-and-paper use, and the instructor whose institution does not possess a moderate-size computer is hard put to find problems which suitably dramatize linear programming without involving a tedious, back-breaking mountain of unenlightening computation.

With the passage of time more colleges and universities will acquire computers and this limitation will grow less critical. But there is another barrier to the recognition and development of linear programming in the undergraduate curriculum. The contexts in which its applications have been most fruitful are unfamiliar to most teachers of mathematics. Though these applications are legion, they have for the most part arisen directly from nonmathematical disciplines in which mathematical techniques have seldom been used in a sophisticated way. (The only exception which comes readily to mind is economics, and even here the spirit of classical mathematical economics has little in common with linear programming.)

We suggest, therefore, that an application in a familiar mathematical setting deserves publicity. Indeed, we offer here an application in polynomial approximation which is admirably suited not only to introduce linear programming in a numerical analysis course, but to solve an important type of problem efficiently.

2. Polynomial approximation. Suppose, from experimental data, one has a finite set of observed points (x_i, y_i) and it is desired to find the polynomial $P_m(x)$ of degree m which is a best fit to these data in some specified sense. The most popular technique is the so-called method of least squares in which $P_m(x)$ is termed a best approximation if it minimizes the expression

$$\sum_i (P_m(x_i) - y_i)^2.$$

Apparently the popularity of the least squares method is due almost entirely to its painless submission to analytic treatment. But in many situations it is obvious that the least squares criterion for best fit is not a good one. Suppose the observed points (x_i, y_i) have the quality that the independent variable x can be measured with great accuracy while the dependent variable is subject to significant not-necessarily-uniform error. It is desired to choose the coefficients of the polynomial $P_m(x)$ so that the quantities $|y_i - P_m(x_i)|$ stay within the region of error. More specifically, one may ask these two questions.

QUESTION 1. *Given $\epsilon_i > 0$ and (x_i, y_i) for $i = 1, \cdots, n$, and nonnegative integers $k_0 < k_1 < \cdots < k_N$, does there exist a polynomial*

$$P(N, k, x) = \sum_{j=0}^{N} a_j x^{k_j}$$

such that $|y_i - P(N, k, x_i)| < \epsilon_i$ for each i?

QUESTION 2. *Given ϵ_i and (x_i, y_i) as above, what is the least N such that Question 1 has an affirmative answer for the polynomial*

$$Q(N, x) = \sum_{j=0}^{N} a_j x^j.$$

Such problems* can be reduced to finding an approximation which is best in the Tchebycheff sense, *i.e.*, which minimizes the maximum error. This is an old problem whose treatment by classical methods is not as succinct as the least squares problem. Among recent contributions we cite the papers of Kelley [3], Motzkin [4] and Selfridge [5]. Particularly relevant to this discussion is Kelley's paper which characterizes the problem of optimum curve fitting to a finite set of points in linear programming terms, apparently for the first time.

It is not our present intent to claim any priority for the discussion which follows, but we are not aware of its existence in the literature. Although Kelley treats Question 2 only for the case of uniform error (all ϵ_i equal) in a manner particularly suited to problems where the number of observed points is very large, the extension of his method to the case of nonuniform error offers no particular difficulty. However, the treatment given below is simpler and more readily accessible to any mathematics major at the upper division level.

3. Linear programming. Let us digress for a moment to discuss the central mathematical problem of linear programming. Consider a linear form

$$z = c_1 x_1 + \cdots + c_n x_n$$

where the variables x_j are constrained by inequalities of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\geqq b_1, \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\geqq b_m, \end{aligned} \qquad (\text{all } x_j \geqq 0).$$

For what values of $x_1, \cdots, x_n$ is z minimized? Since minimizing z is equivalent to maximizing $-z$, there is an equivalent maximization problem. Thus the linear programming problem can be capsulized in the following way: to extremize a linear form subject to linear inequality constraints. Provided the number

* These problems were first brought to my attention by Dr. R. B. Leipnik. I am further indebted to Mr. William Clelland III of the Data Computation Branch, Naval Ordnance Test Station, for computational assistance.

of constraint equations is not too large ($\leqq 255$) the general problem can be solved with present day computing equipment by the simplex method of Dantzig (see [1] and [2]).

4. A linear program for the curve fitting problem. It is sufficient then to reduce the curve fitting problem to an exercise in minimizing a linear form subject to linear constraints. Referring to Question 1, we are given observed points (x_i, y_i) and positive numbers ϵ_i for each $i=1, \cdots, n$. If $k_0<k_1< \cdots <k_N$ are nonnegative integers we wish to know whether there exist coefficients $a_0, a_1, \cdots, a_N$ such that

$$E_i = \eta_i \left| y_i - \sum_{j=0}^{N} a_j x_i^{k_j} \right| < 1 \tag{1}$$

for each i, where $\eta_i=1/\epsilon_i$. If we can compute

$$z = \min_{a} \max_{1\leqq i\leqq n} E_i \tag{2}$$

where the min is taken over all $(N+1)$-tuples $a=(a_0, \cdots, a_N)$, then the existence problem is answered by comparing z and 1.

In order to interpret this as a linear programming problem we define some new variables. For each i and j let u_i, v_i, u_{n+j+1}, v_{n+j+1} be new quantities constrained by

$$\begin{aligned} u_i - v_i &= \eta_i\left(y_i - \sum_{j=0}^{N} a_j x_i^{k_j}\right) && (i = 1, \cdots, n), \\ u_{n+j+1} - v_{n+j+1} &= a_j && (j = 0, \cdots, N), \\ u_i \geqq 0,\ v_i &\geqq 0,\ u_{n+j+1} \geqq 0,\ v_{n+j+1} \geqq 0, \end{aligned} \tag{3}$$

and consider the new problem of determining

$$z' = \min_{a} \max_{1\leqq i\leqq n} (u_i + v_i) \tag{4}$$

subject to

$$\eta_i y_i = u_i - v_i + \sum_{j=0}^{N} \eta_i (u_{n+j+1} - v_{n+j+1}) x_i^{k_j}. \tag{5}$$

Suppose z' is computed and that for some i both u_i and v_i take positive values, say $u_i=\delta_i+c_i$, $v_i=\delta_i$, where $c_i\geqq 0$. It is clear that z' cannot be increased if the new values $u_i=c_i$, $v_i=0$ are assigned, but the constraint (5) is still satisfied. A similar argument applies if $u_i<v_i$ so that a solution to (4) can always be found with $u_i=0$ or $v_i=0$. But in this case $u_i+v_i=E_i$ and we infer that $z'=z$.

Our problem now is to minimize a quantity which is no less than each of n distinct linear forms (of the type u_i+v_i) where the variables are constrained by (3) and (5). Introducing a fictitious slack variable t, we can express the prob-

lem in the following way. Minimize the linear form $w=u_1+v_1+t$, subject to

$$u_1 + v_1 - u_i - v_i + t \geqq 0 \qquad (i = 2, \cdots, n),$$

$$u_i - v_i + \sum_{j=0}^{N} \eta_i x_i^{kj}(u_{n+j+1} - v_{n+j+1}) = \eta_i y_i \qquad (i = 1, \cdots, n),$$

$$u_i \geqq 0,\ v_i \geqq 0,\ t \geqq 0.$$

This is a straightforward linear programming problem and the minimized value of w is precisely the z sought in (2). If $z<1$ then Question 1 has an affirmative answer. To answer Question 2 it suffices merely to repeat the above procedure for $Q(N, x)$ for increasing values of N, commencing with a value so low that a negative answer to Question 1 is assured. However, it seems likely that this brute-force approach is unnecessarily laborious; by appropriately altering the final simplex tableau for a given value of N one may expect to reduce significantly the necessary computation for the next value of N.

5. An alternative formulation. Returning to Section 2, it is sufficient, in order to answer Question 1, to find $\gamma>0$ for which

$$|y_i - P(N, k, x_i)| \leqq \epsilon_i - \gamma. \qquad (i = 1, \cdots, n) \tag{6}$$

An equivalent linear program is to maximize γ subject to

$$\begin{aligned} \gamma - \sum_{j=0}^{N} a_j x_i^{kj} &\leqq \epsilon_i - y_i, \\ \gamma + \sum_{j=0}^{N} a_j x_i^{kj} &\leqq \epsilon_i + y_i, \end{aligned} \tag{7}$$

for $i=1, \cdots, n$. If $\max \gamma>0$ then Question 1 has an affirmative answer.

Every linear programming problem has a dual problem associated with it which is often more readily solved. (See, for example, Chapter V of [2].) The dual of this problem is to minimize μ where

$$\mu = \sum_{i=1}^{n} s_i(\epsilon_i - y_i) + t_i(\epsilon_i + y_i)$$

subject to

$$\sum_{i=1}^{n} (s_i + t_i) = 1,$$

$$\sum_{i=1}^{n} (s_i - t_i)x_i^{kj} = 0, \qquad (j = 0, \cdots, N)$$

$$s_i \geqq 0, \quad t_i \geqq 0.$$

Since $\min \mu = \max \gamma$, the solution of the dual problem produces the answer to

Question 1, and since most linear programming codes will produce the optimum values of the variables of both the primal and dual problems, nothing has been lost. In addition, the final formulation above involves only $N+2$ equations in $2n$ unknowns, so that the dual problem has a smaller matrix, resulting in less computing time. Finally, the number of observed points (x_i, y_i) is now quite unrestricted, for the number of constraint equations depends only on the degree of the approximating polynomial.

6. Numerical examples. We give two examples by way of illustration. For expository simplicity the method of Section 4 is employed in each case.

First, let us find a quadratic expression $a_0+a_1x+a_2x^2$ which is a best fit to the points (1, 2), (2, 5), (3, 3) and (4, 2). Since we seek an optimum solution without regard to prescribed error requirements it is sufficient to take $\epsilon_i=1$. Thus, according to the foregoing, we wish to determine the minimum of

$$w = u_1 + v_1 + t$$

subject to

$$\begin{aligned}
u_1 + v_1 - u_2 - v_2 + t &\geqq 0,\\
u_1 + v_1 - u_3 - v_3 + t &\geqq 0,\\
u_1 + v_1 - u_4 - v_4 + t &\geqq 0,\\
u_1 - v_1 + u_5 - v_5 + u_6 - v_6 + u_7 - v_7 &= 2,\\
u_2 - v_2 + u_5 - v_5 + 2u_6 - 2v_6 + 4u_7 - 4v_7 &= 5,\\
u_3 - v_3 + u_5 - v_5 + 3u_6 - 3v_6 + 9u_7 - 9v_7 &= 3,\\
u_4 - v_4 + u_5 - v_5 + 4u_6 - 4v_6 + 16u_7 - 16v_7 &= 2,
\end{aligned}$$

with all variables nonnegative. A problem of this magnitude is certainly not beyond pencil-and-paper solution. The calculation was performed in short order on an IBM type 704 computer with the following results:

$$\begin{aligned}
w &= 0.75, & u_4 &= 0.75,\\
v_1 &= 0.75, & v_5 &= 0.75,\\
u_2 &= 0.75, & u_6 &= 4.50,\\
v_3 &= 0.75, & v_7 &= 1.00,
\end{aligned}$$

all other variables being zero. In particular, if $\epsilon_i>0.75$ for $i=1, 2, 3, 4$, then the second degree polynomial $P(x)=-0.75+4.5x-x^2$ satisfies $|y_i-P(x_i)|<\epsilon_i$. Moreover, if each $\epsilon_i\leqq 1$ then it is clear by inspection that no polynomial of lower degree is satisfactory.

As a second example we seek a polynomial of the form $a+bx+cx^3$ which is a best fit for the data $(-4, 2)$, $(-2, 1)$, $(1, 0)$ and $(5, 2)$. Again we take $\epsilon_i=1$ so that, as above, we are to minimize $w=u_1+v_1+t$ in nonnegative variables subject to

$$u_1 + v_1 - u_2 - v_2 + t \geqq 0,$$
$$u_1 + v_1 - u_3 - v_3 + t \geqq 0,$$
$$u_1 + v_1 - u_4 - v_4 + t \geqq 0,$$
$$u_1 - v_1 + u_5 - v_5 - 4u_6 + 4v_6 - 64u_7 + 64v_7 = 2,$$
$$u_2 - v_2 + u_5 - v_5 - 2u_6 + 2v_6 - 8u_7 + 8v_7 = 1,$$
$$u_3 - v_3 + u_5 - v_5 + u_6 - v_6 + u_7 - v_7 = 0,$$
$$u_4 - v_4 + u_5 - v_5 + 5u_6 - 5v_6 + 125u_7 - 125v_7 = 2.$$

Another calculation gave the results

$$w = .6296296, \qquad u_4 = .6296296,$$
$$u_1 = .6296296, \qquad u_5 = 1.000000,$$
$$v_2 = .6296296, \qquad v_6 = .3888888,$$
$$v_3 = .6296296, \qquad u_7 = .0185185,$$

all other variables being zero. Thus an optimum solution is $P(x) = 1 - .389x + .0185x^3$.

7. Concluding remarks. Linear programming is no universal balm, a fact not always apparent in some discussions of the subject. Consider this intriguing generalization of the original problem posed in Section 2. Suppose, in addition to the error quantity ϵ_i attributed to the dependent variable, that a significant error δ_i may occur in measuring the independent variable. The analogous problem in this case is to determine whether a polynomial of degree N can be found whose graph intersects each of a collection of similarly oriented rectangles of dimensions $2\delta_i \times 2\epsilon_i$. Here (compare with (1) and (2)) the problem is to compute

$$\min_{a} \min_{|x-x_i| \leq \delta_i} \max_{1 \leq i \leq n} \eta_i \left| y_i - \sum_{j=0}^{N} a_j x^{k_j} \right|.$$

The treatments of Sections 4–5 are now altogether inadequate, for this problem reduces to minimizing a linear form subject to constraints which are exceedingly nonlinear.

The reader will note, however, that the methods developed above permit wider application. Instead of determining $\max \gamma$ in (6) and (7) or z of (2), the terms $x_i^{k_j}$ can be replaced by $f_j(x_i)$ where the functions f_j are completely unrestricted.

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THE BOOLE TABLE GENERALIZED

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In Classroom Notes of this MONTHLY (vol. 67, 1960, pp. 170–172), the writer presented a graphical device for analyzing Boolean functions which were displayed in disjunctive or conjunctive normal form. This device was called a Boole table. The referee indicated how the Boole table could be used to analyze Boolean functions constructed from the operations of product ($\cap$), sum ($\cup$), and complement ($'$), no matter how complexly the elements (x, y, z, $\cdots$) are overlaid by stacks of these operations. The present note briefly describes this more general use of the Boole table.

We shall represent the columns of T's and F's in the conventional truth table of the propositional calculus by bars and the absence of bars. Thus, for three propositional variables, p, q, and r, the conventional truth table and the graphical version are given in Figures 1 and 2.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

FIG. 1

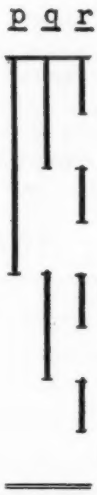

FIG. 2

Due to the isomorphism between the propositional calculus and Boolean algebra, all of the conventional truth table operations can be transformed into equivalent Boolean operations. Logical conjunction ($\cdot$) becomes the Boolean product ($\cap$); logical disjunction ($\vee$) becomes the Boolean sum ($\cup$); logical negation ($\sim$) becomes the Boolean complement ($'$); and so on. The isomorphism

for the graphical version of the truth table is the Boole table.

Let us illustrate the use of the Boole table by analyzing the function, $F=\{[(x\cap y)\cup z']'\cup(x'\cap y)\}$. First, display the Boole table and the function, F, as shown in Figure 3—the reference framework for the Boole table is in a column on the left and the function is written with wide spacing in a row at the

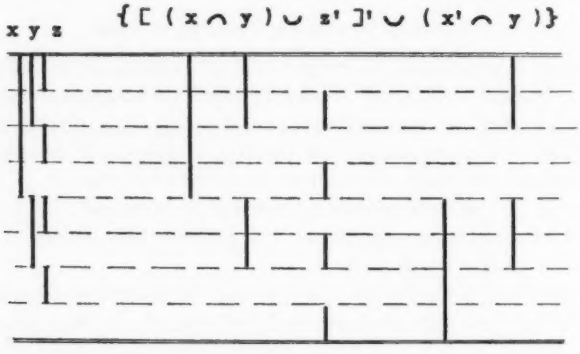

FIG. 3

top. Now one can proceed in this manner. (1) Enter bars for the memberships of sets denoted by single variables. (2) Enter bars for the complements (if any) of (1)-entries. (In Figure 3, steps (1) and (2) have been performed.) (3) Enter bars for every binary operation conjoining preceding entries. (4) Enter bars for the complements of the formulas designated under (3). Continue in this manner until a final column of bars is entered which represents the function as a whole. Figure 4 gives the complete analysis of F, where the final column of bars

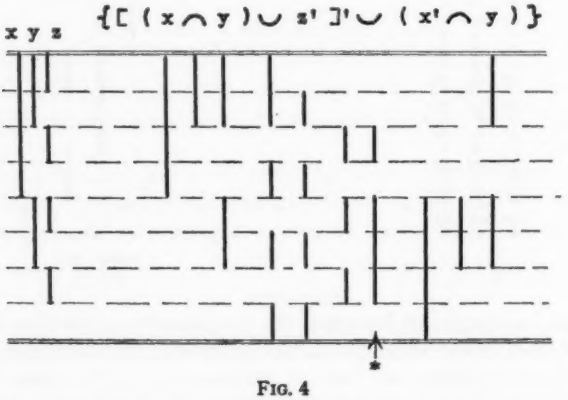

FIG. 4

above the asterisk (*) indicates the membership of the set denoted by the whole

function. The Boole table shows that the function, F, reduces to

$$[(x' \cap y) \cup (y' \cap z)].$$

As a second example, we give the analysis of $G = \{[(x' \cup y')' \cap z]' \cup (x \cup w)'\}'$. It is shown in Figure 5. The function, G, is equivalent to $(x \cap y \cap z)$.

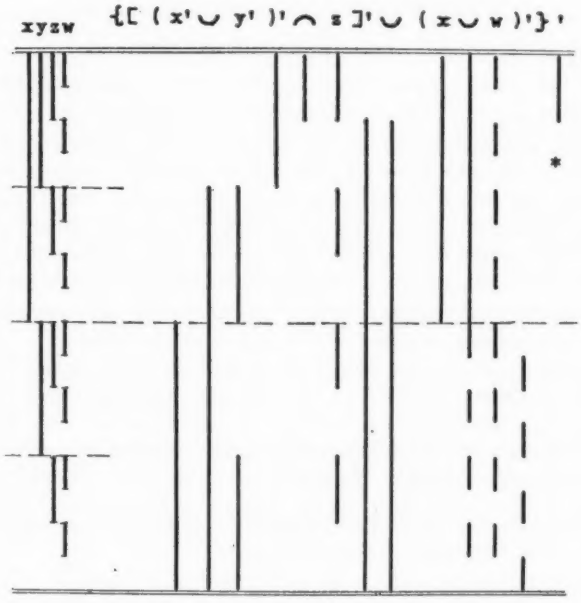

FIG. 5

Where the column of bars indicating the membership of the set designated by the function as a whole is continuous, the function denotes the universe class. In the case where the final column is devoid of bars, the function designates the null class. The function, $H = \{[(x \cup y')' \cap (z' \cup x')'] \cap [(x \cup z) \cup y]'\}$, is an example. Otherwise, the function is "contingent." The functions F and G above illustrate this case.

Turning attention to the isomorphism in the propositional calculus, a brief comment will be sufficient. Take a logical formula expressing an argument. Let it be

$$\{[r \supset (t \vee s)] \cdot [n \supset (s \vee b)] \cdot [\sim s]\} \supset [(\sim t \cdot \sim b) \supset (\sim r \cdot \sim n)],$$

which is the form of the argument in Exercise 24, page 52, of I. M. Copi's *Symbolic Logic* (New York, 1954), where $\cdot$ is logical conjunction, $\vee$ is logical disjunction, $\sim$ is logical negation, and $\supset$ is material implication. In any such

formula, we replace $\supset$ (and $\equiv$, if it occurs) by its definition in terms of conjunction, disjunction, and negation. We can then use the graphical version of the truth table (Fig. 2) and operations analogous to those described above to determine the validity of the argument. Validity requires, of course, a continuous bar down the truth table in the final column for the logical function.

FUBINI IMPLIES LEIBNIZ IMPLIES $F_{yx} = F_{xy}$

R. T. SEELEY, Harvey Mudd College

Three of the most common and important theorems on the interchange of limit operations are the rules for interchanging the order of integration in a repeated integral (Fubini's theorem), differentiating under the integral sign (Leibniz's rule), and taking mixed partial derivatives in either order. This note points out that the first of these leads easily to the other two. This has advantages for the course which aims to prove everything, by replacing two rather difficult arguments with what seem to be simpler ones; and for the course which does not want to handle all complications, by reducing the number of unproved statements. Moreover, the connections between these results seem interesting in themselves, and provide good results for Lebesgue integrals.

The note first assumes a simple form of Fubini's theorem for Riemann integrals (A), uses this to prove Leibniz's rule (B), and uses this in turn to prove one of the stronger forms of the theorem on mixed partials (C). It then states the corresponding results for Lebesgue integrals; the proofs remain the same. R denotes the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$.

(A) *If $f(x, y)$ is continuous on R, then $\int_c^d f(x, y)dy$ is a continuous function of x, $\int_a^b f(x, y)dx$ is a continuous function of y, and*

$$\int_a^b \left[\int_c^d f(x, y)dy \right] dx = \int_c^d \left[\int_a^b f(x, y)dx \right] dy.$$

(B) *If $g(x, y)$ and its partial derivative $g_x(x, y)$ are continuous on R, then $\int_c^d g(x, y)dy$ has a derivative on $a < x < b$, which equals $\int_c^d g_x(x, y)dy$.*

Proof. By the fundamental theorem of calculus (FTC) and Theorem (A),

$$\begin{aligned} \int_c^d g(x, y)dy &= \int_c^d \left[\int_a^x g_x(s, y)ds + g(a, y) \right] dy \\ &= \int_a^x \left[\int_c^d g_x(s, y)dy \right] ds + \int_c^d g(a, y)dy. \end{aligned} \tag{1}$$

By (A), $\int_c^d g_x(s, y)dy$ is continuous, so that the FTC asserts that the expression (1) has the derivative $\int_c^d g_x(x, y)dy$.

(C) *If f_y and f_{yx} are continuous on R, and $f_x(x, c)$ exists for $a < x < b$, then f_x and f_{xy} exist in the interior of R and f_{xy} equals f_{yx}. (Here f_{yx} is $\partial f_y/\partial x$, and f_{xy} is $\partial f_x/\partial y$.)*

Proof. By the FTC, $f(x, y) = \int_c^y f_y(x, t)dt + f(x, c)$. By Theorem (B)

$$f_x(x, y) = \int_c^y f_{yx}(x, t)dt + f_x(x, c). \tag{2}$$

Again by the FTC, the expression on the right in (2) has a derivative with respect to y which equals $f_{yx}(x, y)$.

For the Lebesgue integral, we have the following formulation.

(A′) *If f is integrable on R, then $\int_c^d [\int_a^b f(x, y)dx]dy = \int_a^b [\int_c^d f(x, y)dy]dx$, both repeated integrals existing.*

(FTC′) *If f is integrable on $a<x<b$, then $F(x) = \text{constant} + \int_a^x f(t)dt$ has a derivative $F'(x) = f(x)$ almost everywhere. Any function F of this form is called absolutely continuous.*

(B′) *If g is integrable on R and $G(x, y) = \int_a^x g(t, y)dt + G(a, y)$ for almost every y in $c \leqq y \leqq d$, and $\int_c^d G(a, y)dy$ exists, then $\int_c^d G(x, y)dy$ has a derivative equal to $\int_c^d g(x, y)dy$ for almost every x in $a \leqq x \leqq b$.*

(C′) *If $f(x, y)$, defined on R, is absolutely continuous in y for almost every x, and $f(x, c)$ is absolutely continuous, and f_y is absolutely continuous in x for almost every y, and f_{yx} is integrable on R, then f_x is absolutely continuous in y for almost every x, and $f_{xy} = f_{yx}$ almost everywhere.*

In conclusion, we point out that (B) can be proved by (FTC) and (C), through the device of considering $f(x, y) = \int_c^y g(x, t)dt$; and (*A*) can be gotten in part from (*B*) by showing that the derivatives of $\int_a^x [\int_c^y f(s, t)dt]ds$ equal those of $\int_c^y [\int_a^x f(s, t)ds]dt$. But this sequence appears to be less useful than the one presented.

The proof of (B) given here can also be found in Kaplan, *Advanced Calculus.*

LINE INTEGRALS OF EXACT DIFFERENTIALS

WILLIAM J. FIREY, Washington State University

In this note we prove the following theorem:

If the coefficients $M(x, y)$, $N(x, y)$ of the differential form $Mdx + Ndy$ satisfy $\partial M/\partial y = \partial N/\partial x$ throughout a simply-connected region A, then the line integral

$$\int_{(a_0, b_0)}^{(a_1, b_1)} Mdx + Ndy$$

has the same value for each continuously differentiable arc in A joining (a_0, b_0) to (a_1, b_1).

The usual proof employs Gauss' theorem:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{\partial R} Mdx + Ndy$$

where ∂R is the boundary of a subregion R of A. In the simplest case, R is taken to be a region whose intersections with the lines $x=$constant and $y=$constant are intervals. The extension to more general regions follows by a decomposition argument.

Our proof uses line integrals only. We define the line integral along an arc C as follows. Let C be described parametrically by

$$x = x(t), \qquad y = y(t), \qquad 0 \leqq t \leqq 1,$$

where $(x(0), y(0)) = (a_0, b_0)$, $(x(1), y(1)) = (a_1, b_1)$ and $x(t)$, $y(t)$ are continuously differentiable. Then

$$\int_C Mdx + Ndy = \int_0^1 \left\{M(x(t), y(t)) \frac{dx(t)}{dt} + N(x(t), y(t)) \frac{dy(t)}{dt}\right\} dt.$$

Denote this integral by $J(C)$.

Assume A is convex. Let C_0, C_1 be any two arcs in A joining (a_0, b_0) to (a_1, b_1) described parametrically by $x=x_i(t)$, $y=y_i(t)$, $i=0, 1$, $0 \leqq t \leqq 1$, where $(x_i(0), y_i(0)) = (a_0, b_0)$, $(x_i(1), y_i(1)) = (a_1, b_1)$ and $x_i(t)$, $y_i(t)$ are continuously differentiable. Consider the family of arcs C_ϑ described by

$$(1) \qquad \begin{aligned} x &= (1-\vartheta)x_0(t) + \vartheta x_1(t), \\ y &= (1-\vartheta)y_0(t) + \vartheta y_1(t), \end{aligned} \qquad 0 \leqq t \leqq 1,\ 0 \leqq \vartheta \leqq 1.$$

Plainly each C_ϑ is continuously differentiable and joins (a_0, b_0) to (a_1, b_1). By the convexity of A, C_ϑ lies in A. For short, let the right sides of (1) be denoted by $x_\vartheta(t)$, $y_\vartheta(t)$.

We shall prove, by direct calculation, that

$$dJ(C_\vartheta)/d\vartheta = 0$$

if $\partial M/\partial y = \partial N/\partial x$ in A. From this it follows that $J(C_0) = J(C_1)$, which is the assertion of the theorem. The extension to more general regions can be accomplished by a decomposition argument and will be omitted.*

We have

$$J(C_\vartheta) = \int_0^1 \left\{M(x_\vartheta(t), y_\vartheta(t)) \frac{dx_\vartheta(t)}{dt} + N(x_\vartheta(t), y_\vartheta(t)) \frac{dy_\vartheta(t)}{dt}\right\} dt.$$

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* R. Courant, Differential and Integral Calculus, 2nd ed., New York, 1937, vol. II, p. 362. In that discussion the regions intersected twice by parallels to the axes can be replaced by convex regions. For example, if C_0 and C_1 are in the interior of A, we can determine a subregion A' of A, decomposable into a finite number of triangles, such that C_0 and C_1 lie in A'. If C_0 or C_1 contain boundary points of A, a limit argument must be used.

Therefore

$$\frac{dJ(C_\vartheta)}{d\vartheta} = \int_0^1 \left\{ \left[\frac{\partial M}{\partial x}\frac{\partial x_\vartheta}{\partial \vartheta} + \frac{\partial M}{\partial y}\frac{\partial y_\vartheta}{\partial \vartheta} \right] \frac{dx_\vartheta}{dt} + \left[\frac{\partial N}{\partial x}\frac{\partial x_\vartheta}{\partial \vartheta} + \frac{\partial N}{\partial y}\frac{\partial y_\vartheta}{\partial \vartheta} \right] \frac{dy_\vartheta}{dt} + M\left[\frac{dx_1}{dt} - \frac{dx_0}{dt} \right] + N\left[\frac{dy_1}{dt} - \frac{dy_0}{dt} \right] \right\} dt.$$

From (1):

$$\frac{\partial x_\vartheta}{\partial \vartheta} = x_1 - x_0, \qquad \frac{\partial y_\vartheta}{\partial \vartheta} = y_1 - y_0.$$

Using this and the exactness condition $\partial M/\partial y = \partial N/\partial x$ we obtain

$$\begin{aligned} \frac{dJ(C_\vartheta)}{d\vartheta} &= \int_0^1 \left\{ \left[\frac{\partial M}{\partial x}\frac{dx_\vartheta}{dt} + \frac{\partial M}{\partial y}\frac{dy_\vartheta}{dt} \right] (x_1 - x_0) + M\left[\frac{dx_1}{dt} - \frac{dx_0}{dt} \right] \right. \\ &\quad \left. + \left[\frac{\partial N}{\partial x}\frac{dx_\vartheta}{dt} + \frac{\partial N}{\partial y}\frac{dy_\vartheta}{dt} \right] (y_1 - y_0) + N\left[\frac{dy_1}{dt} - \frac{dy_0}{dt} \right] \right\} dt \\ &= \int_0^1 \frac{d}{dt}\left[(x_1 - x_0)M + (y_1 - y_0)N \right] dt \\ &= \left[(x_1 - x_0)M + (y_1 - y_0)N \right]_{t=0}^{t=1}. \end{aligned}$$

At the upper and lower limits $x_1(t) - x_0(t)$ and $y_1(t) - y_0(t)$ are zero. Hence $dJ(C_\vartheta)/d\vartheta = 0$ and the theorem is proved.

In an elementary course in differential equations, the theorem furnishes a general formula for constructing solutions of exact equations. The foregoing proof allows one to treat this matter without a preliminary discussion of Gauss' theorem.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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OKLAHOMA STATE COMMITTEE FOR THE IMPROVEMENT OF MATHEMATICS INSTRUCTION

JAMES H. ZANT, Oklahoma State University

Introduction. This committee, appointed in 1958 by the Oklahoma Curriculum Improvement Commission of the State Department of Education, consists of teachers of mathematics and mathematics education at all levels, supervisors

and administrators. It is advisory in character and has sponsored meetings, workshops, symposia and experimental teaching in the state. Its primary function is the improvement of mathematics teaching.

Accomplishments. The activities of this Committee have created in the state an interest and concern among both teachers and administrators about a more modern program in mathematics at all levels. Experimental programs have been organized at a number of places over the state. Chief among these experimental programs were the seven centers for teaching the School Mathematics Study Group textbooks in Grades 9, 10 and 11 during 1959–60. This activity, supported by SMSG and supervised by the Committee, involved 23 school systems, and 84 classes with 2,700 high school students. Forty-one high school teachers and 8 college mathematicians worked in the program. A similar program of about the same magnitude has been organized for 1960–61 using SMSG textbooks for Grades 4, 5 and 6. Fifty-two school districts in Oklahoma have bought 13,500 copies of SMSG textbooks Grades 7–12 for use during 1960–61.

The Committee expects to continue its work next year with the primary objective of informing teachers and administrators of promising developments now taking place in the mathematical community with reference to improving mathematics at the school level and encouraging ways of making the necessary knowledge of mathematics available to teachers so that they can teach this exciting material with confidence and enthusiasm. It is expected that a preliminary report will be published soon for use in the state. A report *Tentative Recommendations of the State Mathematics Committee* was published December 1960 and is available from the Oklahoma State Department of Education, Oklahoma City.

UNIVERSITY OF MARYLAND MATHEMATICS PROJECT (JUNIOR HIGH SCHOOL)

The University of Maryland Mathematics Project (Junior High School) is starting its fourth year with special emphasis on inservice programs for elementary teachers and psychological studies in the learning of mathematics. The Maryland experimental texts for grades 7 and 8 are being used by approximately 15,000 junior high school students in all parts of the country during this school year. There will be no further attention given this year to a revision of the texts based on teaching experiences.

An experimental program of inservice education for elementary teachers of mathematics is being conducted by Helen M. Garstens, Associate Director of the project. This program consists of two parts, one of which is a course in mathematics for elementary teachers being offered to 25 teachers selected by the five major school systems in the Washington area. A second part of the program involves a small class in mathematics for elementary school supervisors from seven school systems who attend the class on late Monday afternoons and then offer a somewhat similar course for elementary school teachers in their school systems on another night during the week.

The learning studies in mathematics are to be conducted under the direction of Professor Robert Gagné, Department of Psychology, Princeton University. A full-time psychologist, Noel Paradise, has been employed to work on the Maryland campus. The studies will be based in considerable part on topics selected from the University of Mary-

land experimental courses and will be conducted at the 6th to 8th grade levels. These studies will be related to some of the research being carried on by Professor Gagné at Princeton University.

KENTUCKY CONFERENCE OF COLLEGE SCIENCE AND MATHEMATICS STAFF MEMBERS

The first of the state conferences held as a part of the NASDTEC Teacher Preparation Certification Study (see article by G. S. Young, this MONTHLY, vol. 67, 1960, pp. 792–797) was held in Louisville, Kentucky, on September 23–24. Approximately 150 staff members of the science and mathematics departments of the 34 colleges of the state offering teacher programs in science and mathematics attended. Among the purposes of the conference was to plan how the programs recommended by the NASDTEC regional conferences could become a basis for approval of teacher education programs in Kentucky. A major part of the conference provided for reports from the various curriculum studies in the sciences. Among the consultants were Gerald W. Zacharias, Massachusetts Institute of Technology; E. G. Begle, Yale University; and Arnold Grobman, University of Colorado.

ONTARIO MATHEMATICS COMMISSION

The Ontario Department of Education, together with a number of professional groups, including the Ontario Teachers' Federation, and the Ontario Association of Teachers of Mathematics and Physics, have recently established the Ontario Mathematics Commission. The objectives of the Commission, as stated in the constitution are:

To promote excellence in the teaching of mathematics in the province of Ontario by:

(a) Undertaking curriculum research aimed at keeping the province abreast of the best contemporary practice and circulating such information to interested bodies;
(b) Encouraging the production of experimental teaching material in mathematics, and seeking the cooperation of the Department of Education in testing such material in the schools of the province;
(c) Cooperating with the universities, the Ontario Teachers' Federation, and other appropriate bodies, in providing courses which will enable teachers to improve their qualifications and keep abreast of current experimentation in curriculum changes and in teaching techniques;
(d) Establishing scholarships and prizes for teachers and students of mathematics;
(e) Maintaining communication with similar groups in other provinces and countries; and
(f) Such other means as the Commission may deem necessary.

An article, *Reforming the high school mathematics curriculum*, by A. J. Coleman appears in the Canad. Math. Bull., vol. 3, 1960. The article reviews some of the considerations which led to the formation of the Commission and also gives brief outlines for mathematics courses, grades 9 through 12. It is indicated in the article that anyone wishing to keep abreast of developments in Ontario may write to the Ontario Mathematics Commission, 1260 Bay Street, Toronto 5, Ontario, and ask to be put on the mailing list.

Careers in Mathematics

A short pamphlet on choosing a career in mathematics has been prepared under the auspices of the Conference Board of the Mathematical Sciences. The pamphlet is a reprint from the *Mathematics Teacher*, May 1960. One section is on teaching careers in mathematics and the second section on industrial and government careers in mathematics. Copies of the reprint may be obtained by writing to Dr. G. Baley Price, Executive Secretary, Conference Board of the Mathematical Sciences, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1446. *Proposed by David Bickerstaff, University of Mississippi*

"How about telling me confidentially the secret order of the five beauties to be featured in this year's Annual?" I proposed to the editor. She, of course, refused, but agreed to pass judgment on my guess. "Is it *A-B-C-D-E*?" I asked.

"You are most skillful at being wrong," she chided. "You not only got each person out of her true position but, furthermore, not one in your ranking followed correctly her immediate predecessor."

"Well, then, is it *D-A-E-C-B*?" I asked.

"Now you are improving," she encouraged cautiously. "You have two in proper position and you have two following correctly their immediate predecessors."

After a little figuring I then told her the correct order, and she swore me to secrecy. What is the correct order?

E 1447. *Proposed by Walter Bluger, Dominion Bureau of Statistics, Ottawa, Canada*

Construct a triangle given R, r, h_a.

E 1448. *Proposed by C. B. Grosch, General Mills, Inc., Minneapolis, Minnesota*

Show that any plane section of an oblate spheroid, not perpendicular to the axis of the spheroid, is an ellipse with major axis parallel to the equatorial plane of the spheroid and with minor axis (or minor axis extended) intersecting the axis of the spheroid.

E 1449. *Proposed by C. S. Patlak, Department of Health, Education, and Welfare, Bethesda, Maryland*

Assume that (1) A_i, B_i, C_i, D_i $(i=1, \cdots, n)$ are all positive, (2) $\sum A_i \geqq \sum C_i$, (3) $A_i - C_i = B_i - D_i$ $(i = 1, \cdots, n)$. $P_i = A_iB_i/C_iD_i$ and $P = \max(P_1, \cdots, P_n)$. Prove that $(\sum A_i)(\sum B_i)/(\sum C_i)(\sum D_i) \leqq P$.

E 1450. *Proposed by Lawrence Shepp, Princeton University*

If a_n, $b_n > 0$, $a_n \downarrow 0$, then $\sum_{n=1}^{\infty} (a_n - a_{n+1}) \sum_{j=1}^{n} b_j = \sum_{n=1}^{\infty} a_n b_n$.

SOLUTIONS

Sum of Two Squares

E 1416 [1960, 474]. *Proposed by Leonard Cohen, City College of New York*

Prove that $x^2+y^2=z^n$, where n is a positive integer, always has nonzero integral solutions.

I. *Solution by I. D. Ruggles, San Jose State College.* If n is odd, let $n=2k+1$, $k=0, 1, 2, \cdots$; then $(2^k)^2+(2^k)^2=2^{2k+1}$. If n is even, let $n=2k+2$, $k=0, 1, 2, \cdots$; then $(4\cdot 5^k)^2+(3\cdot 5^k)^2=5^{2k+2}$.

II. *Solution by J. W. Ellis, Louisiana State University in New Orleans.* If (a, b, c) is any Pythagorean triple, then $(ac^{n-1})^2+(bc^{n-1})^2=(c^2)^n$.

III. *Solution by Sidney Kravitz, Dover, N. J.* An infinite number of solutions exist for both $n=1$ and $n=2$. For every solution $x^2+y^2=z^n$, there exists a solution $(zx)^2+(zy)^2=z^{n+2}$.

IV. *Solution by William Becker, New York, N. Y.* Let $z=a^2+b^2$. The conclusion follows from the fact that the product of the sum of two squares is again the sum of two squares.

V. *Solution by L. R. Ford, Charlottesville, Va.* Let u be any complex integer whose nth power, $x+iy$, is neither real nor pure imaginary, and let $u\bar{u}=z$; then $u^n\bar{u}^n=x^2+y^2=z^n$.

VI. *Solution by J. R. Trollope and Walter Zayachkowski, University of Alberta.* Choose for z a prime $p\equiv 1 \pmod 4$. Since every integer which is the product of such primes can be expressed as the sum of two integral squares, it follows that $x^2+y^2=z^n$ always has nonzero integral solutions.

Also solved by Ray Authement, Leon Bankoff, Robert Bart, D. A. Breault, Brother Joseph Heisler, J. L. Brown, Jr., James Burling, Marcus Charles, P. R. Chernoff, Richard Cottle, C. H. Cunkle, Monte Dernham, Gus Di Antonio, F. J. Duarte, G. W. Erwin, Jr., J. C. Ferguson, R. D. Freeman, Jr., B. E. Fristedt, Michael Goldberg, L. D. Goldstone, R. Gramann, Bernard Greenspan, Corinne Hattan, Margaret Herzog, Vern Hoggatt, J. E. Homer, Jr., A. S. Howard, J. A. H. Hunter, Ronald Jacobowitz, Gerald Janusz, J. Jordan, Ray Jurgensen, Irving Katz, P. G. Kirmser, M. S. Klamkin, Donald Knuth, J. D. E. Konhauser, Robert Kruse, Harry Langman, A. T. Lauria, Aaron Lieberman and Sheldon Weinberg (jointly), J. R. Lux, C. R. MacCluer, Peter Marks, D. C. B. Marsh, N. S. Mendelsohn, M. V. Mielke, J. B. Muskat, Judith Ng, Thomas O'Brien, D. J. Persico, R. A. Phillips, C. F. Pinzka, T. J. Robinson, Jonathan Robison, Ray Rogers, Norman Schaumberger, Allen Shields, Jack Silver, D. L. Silverman, M. L. Slater, T. H. Slook, Samuel Stern, E. E. Strock, Eric Sturley, J. R. Sullivan, Wu Ta-Sun, L. Thomas, Allan Trojan, Chih-yi Wang, W. C. Waterhouse, J. S. White, and the proposer. Late solution by Guy Torchinelli.

Editorial Note. Solution II leads to the two-parameter solution $x=2uv(u^2+v^2)^{n-1}$, $y=(u^2-v^2)\times(u^2+v^2)^{n-1}$, $z=(u^2+v^2)^2$; Solution I shows that this does not include all solutions. Solution IV is based on the identity $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$. The basic fact used in Solution IV is found in most texts on the theory of numbers, see, *e.g.*, Jones, *The Theory of Numbers*, Th. 6.6a, p. 133. Th. 6.6b, p. 135 of this reference, solves Problem E 1416 by giving necessary and sufficient conditions for a number to be the sum of two squares.

A Curious Sequence

E 1417 [1960, 474]. *Proposed by Robert Hartop, Los Angeles, California*

Given a unit circle with point P on its circumference and a distance d_1, $0<d_1<2$. With center at P, construct the circle of radius d_1 to cut the given circle in two points. If d_2 is the distance between these points, construct a new circle of radius d_2 with center at P and so obtain $d_3, d_4, \cdots, d_n, \cdots$. Find $\lim_{n\to\infty} d_n$.

Solution by D. C. B. Marsh, Colorado School of Mines. Recursively, $d_n = d_{n-1}(4-d_{n-1}^2)^{1/2}$, which transforms into $|2 \sin \phi_n| = |2 \sin 2\phi_{n-1}|$ by setting $d_j = |2 \sin \phi_j|$ for all j. The latter implies $\phi_n = k_n\pi \pm 2\phi_{n-1}$ (with k_n an arbitrary integer) and has as solution $\phi_n = M\pi \pm 2^{n-1}\phi_1$ (M an arbitrary integer). If a limit exists ($d_n \to d$), then $d=0$ or $\sqrt{3}$, so that $\phi_n \to \phi = k\pi$ or $k\pi \pm \pi/3$ (which may be written as $K\pi/3$) for all $n \geqq$ some positive integer p. The converse also holds, so that $\lim_{n\to\infty} d_n$ exists if and only if d_1 is of the form $d_1 = |2 \sin(K\pi/3\cdot 2^p)|$ for some positive integral p and K, in which case the limiting value is $|2 \sin(K\pi/3)|$.

Also solved by Robert Bart, Frederick Cunningham, Jr., J. A. Faucher, Michael Goldberg, R. Gramann, R. E. Greenwood, J. O. Herzog, Vern Hoggatt and I. D. Ruggles (jointly), A. R. Hyde, Gerald Janusz, Ray Jurgensen, L. M. Kaplan, P. G. Kirmser, J. D. E. Konhauser, William Lopez, Peter Marks, D. A. Moran, D. E. Robison and E. M. Scheuer (jointly), G. B. Robison, Ray Rogers, L. L. Sleizer, L. Thomas, J. S. White, and the proposer. Most of these solutions were incomplete.

Editorial Note. If S and T are the sets of values of d_1 which lead to the limits 0 and $\sqrt{3}$ respectively, it can be shown that S and T are each denumerable and dense.

Interesting is the case where d_1 is chosen as a side of a regular n-gon. If $n=2^k$, then $\lim d_n = 0$; if $n = 3\cdot 2^k$, then $\lim d_n = \sqrt{3}$; in all other cases the sequence $\{d_n\}$ becomes periodic.

Weakened Hypotheses

E 1418 [1960, 474]. *Proposed by R. C. Buck, Institute for Defense Analyses, Princeton, New Jersey*

In *Comptes Rend.* (1945) vol. 62, pp. 95–97, there appears the theorem: If $f(x, y)$ has partial derivatives of first and second orders, and if $f(a, b)+f(b, c) = f(a, c)$ for all a, b, c, then there is a function ϕ such that $f(x, y) = \phi(x) - \phi(y)$. Can these hypotheses be weakened?

Solution by W. C. Waterhouse, Harvard University. We need only $f(a, b) + f(b, k) = f(a, k)$ for a fixed k, as can be seen by putting $\phi(x) = f(x, k)$.

Also solved by Alan Beal, Robert Breusch, P. R. Chernoff, H. E. Chrestenson, R. B. Deal, N. J. Fine, Don Freeman and Fred Gilman (jointly), Michael Goldberg, Gerald Janusz, A. F. Kaupe, Jr., P. G. Krimser, M. S. Klamkin, C. R. MacCluer, Peter Marks, D. C. B. Marsh, E. J. Mickle, Ray Rogers, Jack Silver, Allan Trojan, J. S. White, Albert Wilansky, J. E. Wilkins, Jr., R. J. Wisner, and the proposer.

Deal showed that if S is any nonempty set and f is a function on $S\times S$ to a group G such that for all a, b and one particular c in S, $f(a, b)f(b, c) = f(a, c)$, then there exists a function ϕ on S to G such that $f(x, y) = \phi(x)[\phi(y)]^{-1}$.

An Oversight in the Collected Papers of Ramanujan

E 1419 [1960, 474]. *Proposed by C. C. Yalavigi, Government College, Mercara, India*

In the *Collected Papers of Srinivasa Ramanujan*, edited by G. H. Hardy, Seshu Aiyar, and B. M. Wilson (1928), appears (p. 334, Question 1076 (XI, 199)) the following:

Show that

(i) $$[7(20)^{1/3} - 19]^{1/3} = (5/3)^{1/3} - (2/3)^{1/3},$$

(ii) $$[4(2/3)^{1/3} - 5(1/3)^{1/3}]^{1/6} = (4/9)^{1/3} - (2/9)^{1/3} + (1/9)^{1/3}.$$

Show that these relations are incorrect, and that to correct them we must interchange the exponents 1/8 and 1/6 appearing in the left-hand sides.

Solution by Robert Bart, Michigan College of Mining and Technology. (i) Set $x=(5/3)^{1/3}$, $y=(2/3)^{1/3}$. Then

$$\begin{aligned}(x-y)^6 &= (x^6 - 20x^3y^3 + y^6) + x^2y(15y^3 - 6x^3) + xy^2(15x^3 - 6y^3)\\ &= -19 + 21xy^2 = -19 + 7(20)^{1/3}.\end{aligned}$$

Therefore $[7(20)^{1/3} - 19]^{1/6} = x - y = (5/3)^{1/3} - (2/3)^{1/3}$.

(ii) Set $x = (2/3)^{1/3}$, $y=-1/3)^{1/3}$. Then

$$\begin{aligned}(x+y)^3 &= x^2(x^6 + 56x^3y^3 + 28y^6) + xy(8x^6 + 70x^3y^3 + 8y^6)\\ &\quad + y^2(28x^6 + 56x^3y^3 + y^6)\\ &= 16x^2 + 20xy + 25y^2,\end{aligned}$$

whence $(x+y)^3(4x-5y) = (4x)^3 - (5y)^3 = 128/3 - 125/3 = 1 = x^3+y^3$.
Therefore

$$\begin{aligned}[4(2/3)^{1/3} - 5(1/3)^{1/3}]^{1/3} &= (4x - 5y)^{1/3} = (x^3 + y^3)/(x + y)\\ &= x^2 - xy + y^2 = (4/9)^{1/3} - (2/9)^{1/3} + (1/9)^{1/3}.\end{aligned}$$

Also solved by A. N. Aheart, Leon Bankoff, D. A. Breault, Robert Breusch, Marcus Charles, F. J. Duarte, J. A. Faucher, Michael Goldberg, L. D. Goldstone, Corinne Hattan, J. E. Homer, Jr., Gerald Janusz, Sidney Kravitz, Peter Marks, D. C. B. Marsh, D. J. Persico, D. L. Silverman, C. L. Sterling, W. B. Stovall, Jr., G. C. Thompson, Walter Zayachkowski, and the proposer.

A Property of the Nine-Point Center

E 1420 [1960, 474]. *Proposed by the late Victor Thébault, Tennie, Sarthe, France*

Let A', B', C' (A'', B'', C'') be the centers of squares described exteriorly (interiorly) on the sides BC, CA, AB of a triangle ABC. Show that the radical center of the circles $A(A')$, $B(B')$, $C(C')$ ($A(A'')$, $B(B'')$, $C(C'')$) coincides with the nine-point center of triangle ABC.

Solution by Leon Bankoff, Los Angeles, Calif. Let N denote the nine-point

center, P the orthogonal projection of N upon BC, Q the midpoint of BC, and R the foot of the altitude from A. It is sufficient to show that the powers of N with respect to any pair of circles of the same triad are equal; for example, that $(NB)^2-(NC)^2=(BB')^2-(CC')^2=(BB'')^2-(CC'')^2$. Now

$$\begin{aligned}(NB)^2-(NC)^2 &= (BP)^2-(PC)^2=(BP+PC)(BP-PC)\\ &= a(BQ-RC)=a^2/2-ab\cos C,\\ (BB')^2-(CC')^2 &= [a^2+b^2/2-ab\sqrt{2}\cos(C+45°)]\\ &\quad -[a^2+c^2/2-ac\sqrt{2}\cos(B+45°)]\\ &= ac\cos B-ab\cos C+(b^2-c^2)/2.\end{aligned}$$

Also

$$\begin{aligned}(BB'')^2-(CC'')^2 &= [a^2+b^2/2-ab\sqrt{2}\cos(C-45°)]\\ &\quad -[a^2+c^2/2-ac\sqrt{2}\cos(B-45°)]\\ &= ac\cos B-ab\cos C+(b^2-c^2)/2.\end{aligned}$$

Then, by virtue of the cosine law, we have

$$(NB)^2-(NC)^2=(BB')^2-(CC')^2=(BB'')^2-(CC'')^2.$$

Consequently N coincides with the radical center common to each triad of circles.

Also solved by L. D. Goldstone, Gerald Janusz, Peter Marks, D. C. B. Marsh, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4941. *Proposed by Leonard Carlitz, Duke University*

Bateman's polynomial $F_n(z)$ is defined by

$$F_n(z)={}_3F_2\left[\begin{matrix}-n, & n+1, & \frac{1}{2}(1+z);\\ 1, & 1 & \end{matrix}\right]$$

in the usual notation for generalized hypergeometric functions. Now if $p=2n+1$ is prime, show that

$$F_n(0) = \begin{cases} 0 \pmod{p} & (n \text{ odd}), \\ 4a^2 \pmod{p} & (n \text{ even}), \end{cases}$$

where a^2 is the odd square in the decomposition $p=a^2+b^2$.

This result is analogous to that of problem 4628 [1956, 348].

4942. *Proposed by R. H. Bruck, University of Wisconsin*

Let p be an arbitrary but fixed rational prime. For each positive integer n define the permutation T_n to be the following product of p^{n-1} disjoint cycles of length p:

$$T_n = \prod_i (i, i+p^{n-1}, i+2p^{n-1}, \cdots, i+(p-1)p^{n-1}), \quad i = 1, 2, \cdots, p^{n-1}.$$

Let $G(n)$ be the permutation group generated by $T_1, T_2, \cdots, T_n$. Also define G to be the union of all the groups $G(n)$. Prove:

(a) $G(n)$ is a maximal p-subgroup of the symmetric group on $1, 2, \cdots, p^n$.

(b) Every finite p-group is isomorphic to at least one subgroup of G.

(c) $G(n)$ is nilpotent of class p^{n-1},

(d) G is a locally finite p-group such that $(G, G)=((G, G), G)$.

(e) G is a maximal p-subgroup of the group of all "finite" permutations of the positive integers.

4943. *Proposed by D. J. Newman, Yeshiva University*

Let A and B be two convex plane sets, both symmetric about the origin. Suppose that $z\in A$, $\zeta\in A$ implies that either $z+\zeta\in B$ or $z-\zeta\in B$. Show that this persists for n vectors, *i.e.*, $z_i\in A$, $i=1, 2, \cdots, n$ implies $z_1\pm z_2\pm \cdots \pm z_n \in B$ for some choice of $\pm$ signs.

4944. *Proposed by Morris Newman, National Bureau of Standards*

Prove that $p(n)$, the total number of partitions of n, is odd for infinitely many values of n and even for infinitely many.

4945. *Proposed by N. R. Riesenberg, Brooklyn College*

If $F_n(x)$ is the coefficient of z^n in the expansion of $2hze^{xz}/(e^{hz}-e^{-hz})$ in ascending powers of z so that $F_0(x)=1$, $F_1(x)=x$, $F_2(x)=(3x^2-h^2)/6$, etc., show that:

(1) $F_n(x)$ is a homogeneous polynomial of degree n in x and h.

(2) $dF_n(x)/dx=F_{n-1}(x) \quad n\geqq 1$.

(3) $\int_{-h}^{h} F_n(x)dx=0 \quad n\geqq 1$.

(4) If $y=a_0F_0(x)+a_1F_1(x)+a_2F_2(x)+\cdots$, where a_i are real constants, then the mean value of d^ry/dx^r in the interval $-h\leqq x\leqq h$ is a_r.

4946. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Let $S_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$. Sum $\sum_1^\infty S_n/n!$

SOLUTIONS

A Function Regular Everywhere but at the Composite Integers

4555 [1953, 555]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

If $\tanh u=\tanh^2(\pi\sqrt{z}\sin\theta)$ and $\tanh v=\tan^2(\pi\sqrt{z}\cos\theta)$, then the function* $F(z)=\int_0^{\pi}\coth(u+v)d\theta+4\cot\pi z$ has simple poles at the composite integers and is regular everywhere else.

Solution by the proposer. The expansion

$$f(z)=\pi z\cot\pi z-1=\sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2} \tag{1}$$

yields $f(z/m)=\sum_{n=1}^{\infty}2z^2/(z^2-m^2n^2)$. If we sum on m we find that the denominator z^2-N^2 occurs once for each divisor of N. That is,

$$g(z)=\sum_{m=1}^{\infty}f(z/m)=\sum_{n=1}^{\infty}\frac{2z^2d(n)}{z^2-n^2},$$

where $d(n)$ is the number of divisors of n, counting n and 1, but $d(1)=1$.

From (1) it follows that

$$f(z)=-2\sum_{k=1}^{\infty}H_{2k}z^{2k},\quad\text{where}\quad H_{2k}=\sum_{n=1}^{\infty}n^{-2k}. \tag{2}$$

(See Franklin, *Treatise on Advanced Calculus*, p. 474.) Therefore, by inspection,

$$g(z)=-2\sum_{k=1}^{\infty}H_{2k}^2z^{2k}. \tag{3}$$

If $h(z)=\sum a_nz^n$, then for small $|z|$

$$\frac{1}{2\pi}\int_0^{2\pi}h(e^{i\theta}\sqrt{z})h(e^{-i\theta}\sqrt{z})d\theta=\sum a_n^2z^n.$$

Applying this to $h(z)=\pi z\cot\pi z$ in (2) we get

$$1+\sum_{k=1}^{\infty}(2H_{2k})^2z^{2k}=\frac{\pi z}{2}\int_0^{2\pi}\frac{1+\tan^2c\tanh^2s}{\tan^2c+\tanh^2s}d\theta, \tag{4}$$

where $c=\pi z^{1/2}\cos\theta$, $s=\pi z^{1/2}\sin\theta$. Since

$$\coth(u+v)=\frac{1+\tanh u\tanh v}{\tanh u+\tanh v},$$

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* The coefficient 4 was misprinted in the original proposal.

the choice $\tanh u = \tanh^2 s$, $\tanh v = \tan^2 c$ in (4) gives

$$2g(z) = 1 - \pi z \int_0^{\pi} \coth(u + v)\,d\theta$$

when we observe that the integrand is even. This result and (1) give the explicit formula

$$\int_0^{\pi} \coth(u + v)\,d\theta + 4 \cot \pi z = \frac{5}{\pi z} + \frac{4z}{\pi} \sum_{n=1}^{\infty} \frac{d(n) - 2}{n^2 - z^2},$$

and, hence, the result follows. (We must take 0 and 1 as composite, and not discriminate against the negatives of composite integers.)

Derivatives of a Composite Function

4782 [1958, 212]. *Proposed by V. F. Ivanoff, San Carlos, Calif.*

Given a composite function $F(x) \equiv f[g(x)]$. Denoting the nth derivative of $f(g)$ by $D^n f$, and the derivatives of $g(x)$ by $g', g'', \cdots, g^{(n)}$, show that

$$(1) \qquad F^{(n)}(x) = \begin{vmatrix} g' & g'' & g''' & g^{iv} & \cdots & g^{(n)} \\ -1 & g'D & 2g''D & 3g'''D & \cdots & \binom{n-1}{1} g^{(n-1)} D \\ 0 & -1 & g'D & 3g''D & \cdots & \binom{n-1}{2} g^{(n-2)} D \\ 0 & 0 & -1 & g'D & \cdots & \binom{n-1}{3} g^{(n-3)} D \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots -1 & & g'D \end{vmatrix} Df.$$

Solution by Frank Schmittroth, Oregon State College. Expand the determinant by minors, using elements of the last row:

$$F^{(n)}(x) = g'DF^{(n-1)}(x) + \begin{vmatrix} g' & g'' & \cdots & g^{(n-2)} & g^{(n)} \\ -1 & g'D & \cdots & \binom{n-3}{1} g^{(n-3)} D & \binom{n-1}{1} g^{(n-1)} D \\ 0 & -1 & \cdots & \binom{n-3}{2} g^{(n-4)} D & \binom{n-1}{2} g^{(n-2)} D \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & \binom{n-1}{n-2} g''D \end{vmatrix} Df.$$

Continuing in like manner:

$$F^{(n)}(x) = g'DF^{(n-1)}(x) + \binom{n-1}{n-2} g''DF^{(n-2)}(x) + \cdots + \binom{n-1}{1} g^{(n-1)}DF'(x) + g^{(n)}Df,$$

where $F^{(n-1)}(x)$, $F^{(n-2)}(x)$, $\cdots$, $F'(x)$ can be written as determinants. From $F(x)=f[g(x)]$ obtain at once $F'(x) = (df/dg)g'(x) = g'Df$, which agrees with (2) for $n=1$.

The argument now proceeds by induction. Assume (2) to be true for all $n \leqq N$. Replace n in (2) by N, differentiate both sides with respect to x, and group like terms:

$$\begin{aligned} F^{(N+1)}(x) = {} & g'DF^{(N)}(x) + \binom{N-1}{N-2} g''DF^{(N-1)}(x) + \binom{N-1}{N-3} g'''DF^{(N-2)}(x) \\ & + g''DF^{(N-1)}(x) + \binom{N-1}{N-2} g'''DF^{(N-2)}(x) \\ & + \cdots + \binom{N-1}{1} g^{(N-1)}DF''(x) + \binom{N-1}{0} g^{(N)}DF'(x) \\ & + \cdots + \binom{N-1}{2} g^{(N-1)}DF''(x) + \binom{N-1}{1} g^{(N)}DF'(x) + g^{(N+1)}Df. \end{aligned} \tag{3}$$

Making use of the fact,

$$\binom{N-1}{N-k+1} + \binom{N-1}{N-k} = \binom{N}{N-k+1},$$

and the fact (easily proved) that the order of differentiation with respect to g and differentiation with respect to x may be reversed, equation (3) becomes

$$F^{(N+1)}(x) = g'DF^{(N)}(x) + \binom{N}{N-1} g''DF^{(N-1)}(x) + \binom{N}{N-2} g'''DF^{(N-2)}(x) + \cdots + \binom{N}{2} g^{(N-1)}DF''(x) + \binom{N}{1} g^{(N)}DF'(x) + g^{(N+1)}Df,$$

which is the same as (2) with n replaced by $N+1$. Thus (2) is proved and the equivalent relation (1) is established.

Other formulas for the nth derivative of a compound function are given in: Arnold Dresden, [1943, 9]; M. McKiernan, [1956, 331]; also E. P. Adams, *Smithsonian Misc. Collection*, 74 (1922), No. 1, p. 157; I. M. Ryzhik, *Tables of Series, Products and Integrals*, 3d ed., Berlin (1957), p. 20.

Diophantine Equations

4832 [1959, 147; 1959, 924]. *Proposed by A. Oppenheim, University of Malaya in Kuala Lumpur*

Find all integral solutions of the following Diophantine equations

(A) $$ax^2 + ay^2 + z^2 - 2axyz - 1 = 0 \qquad (a = 1, 2, \cdots),$$

(B) $$9x^2 + 25y^2 + 49z^2 - 210xyz - 1 = 0,$$

(C) $$9x^2 + 25y^2 + 4z^2 - 60xyz - 1 = 0.$$

II. *Solution by the proposer.* The equation (A) for the case $a=1$ was solved in the proposer's note, *On the Diophantine equation* $x^2+y^2+z^2+2xyz=1$. (This MONTHLY, LXIV (1957), 101–103.) A similar method applies here, and we have the

THEOREM. All integral solutions (x, y, z) of (A) (except the trivial solutions $(0, 0, \pm 1)$) are found uniquely by the following rule: (i) Take arbitrary non-negative integers p, q, r, with greatest common divisor unity, with p, q odd, and such that one of them is equal to the sum of the other two; (ii) define θ by $\cosh \theta = a^{\frac{1}{2}}u$, where u is any positive integer; then the numbers x, y, z are defined by

$$a^{1/2}x = \pm \cosh p\theta, \qquad a^{1/2}y = \pm \cosh q\theta, \qquad z = \pm \cosh r\theta,$$

where the product of the ambiguous signs is $+1$.

That z is an integer is clear since $\cosh r\theta$ is an integral polynomial in even powers of $a^{\frac{1}{2}}u = \cosh \theta$; and x, y are integers since each of $\cosh p\theta$, $\cosh q\theta$ is $a^{\frac{1}{2}}u$ times an integral polynomial in even powers of $a^{\frac{1}{2}}u = \cosh \theta$. For example, $p=5$, $q=3$, $r=2$ give the solutions:

$$x = 16a^2u^5 - 20au^3 + 5u, \qquad y = 4au^3 - 3u, \qquad z = 2au^2 - 1.$$

The proof of the theorem may be obtained by the method of descent used in the note cited above. Suppose a solution exists such that $\min(|x|, |y|, |z|) > 1$. We develop a triad with a smaller minimum and, hence, after a finite number of steps, a triad with $\min = 1$ or 0.

If (x, y, z) is a solution, so also are (x', y, z), (x, y', z) and (x, y, z') (called associated solutions), where

$$x' = 2yz - x, \qquad y' = 2zx - y, \qquad z' = 2axy - z.$$

For if $\phi(x) = 0$, where $\phi(X) = aX^2+ay^2+z^2-2aXyz-1$, so also is $\phi(x') = 0$, since $x+x' = 2yz$. Similarly for y' and z'.

Without loss of generality we take $|x| \geqq |y|$. Suppose $|x| \geqq |y| \geqq |z| > 1$. Then the associated solution (x', y, z) is such that $|y| \geqq |z| \geqq |x'|$, as is easily shown. Similarly in case $|x| \geqq |z| \geqq |y| > 1$ or $|z| \geqq |x| \geqq |y| > 1$. In all cases, therefore, after a finite number of steps we reach a solution in which $\min(|x|, |y|, |z|) = 1$ or 0.

If x or $y=0$, we get simply $(0, 0, \pm 1)$. If $z=0$, there is no solution. If $x=\pm 1$ (or $y=\pm 1$), descent yields $x=\pm 1$, $y=\pm 1$, $z=\pm 1$. Finally, if $z=\pm 1$, then $x=\pm y(|x| \geqq 1)$.

Induction on the basic triads $(u, u, 1)$, $(u, -u, 1)$, with u a positive integer, leads to the theorem as stated.

A similar method applies to (B) and to other equations of the form

$$(*) \qquad a^2x^2 + b^2y^2 + c^2z^2 - 2abcxyz - 1 = 0,$$

where a, b, c, are integers with greatest common divisor unity. In fact, putting $X=ax$, $Y=by$, $Z=cz$ we have the case already discussed. The previous solution then applies except that we must have finally X, Y, Z divisible by a, b, c, respectively.

In particular for equation (B) it is not difficult to find the conditions under which the divisibility requirements on X, Y, Z are met and thus determine that all solutions are given by the rule: (i) p, q are odd integers, $r \equiv 2 \pmod 4$; coprime; one equal to the sum of the other two; (ii) $\cosh \theta = 105t \pm 30$, with t integral; then the solutions are x, y, z where

$$3x = \pm \cosh p\theta, \qquad 5y = \pm \cosh q\theta, \qquad 7z = \pm \cosh r\theta$$

with product of ambiguous signs equal to $+1$.

For the more general case, it is not difficult to prove the following result: The necessary and sufficient condition that solutions (in fact, infinitely many solutions) of (*) exist is that at least one of a, b, c is of the form $8k \pm 1$.

Thus there are no integers satisfying (C), as has already been shown in I.

Bounded Functions

4879 [1959, 921]. *Proposed by L. J. Wallen, Massachusetts Institute of Technology*

Let f be a measurable real function defined for all real x, and let G be a continuous real function of two variables. Show that if

$$|f(x+y)| \leqq G(f(x), f(y))$$

for all x and y, then f is bounded on bounded sets.

Solution by D. J. Newman, Brown University. Since f is measurable, the function h defined by $h(x) = |f(x)| + |f(-x)|$ is measurable. Thus there exists some set S of positive measure on which h is bounded. Thus the hypotheses imply that $f(x-y)$ is bounded for all x, y in S. Now it is well known, since S has positive measure, that the set of all $x-y$ with x and y in S is a neighborhood of the origin. Hence $f(z)$ is bounded for all z in some neighborhood U of the origin. Thus, by applying induction to the hypotheses, it follows that, for an arbitrary positive integer n, $f(nz)$ is bounded for all z in U. That is, f is bounded on any bounded set.

Also solved by the proposer.

Integral Equation

4885 [1960, 87]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass., and D. J. Newman, Brown University*

Determine the unique solution of the integral equation

$$F(x_1, x_2, \cdots, x_n) = 1 + \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} F(y_1, y_2, \cdots, y_n)dy_1dy_2\cdots dy_n.$$

(The uniqueness when $n=2$ was one of the problems in the 1958 Putnam competition.)

Solution by P. G. Rooney, University of Toronto. Let $u_0=1$, and

$$u_{r+1}(x_1, \cdots, x_n) = 1 + \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} u_r(y_1, \cdots, y_n)dy_1\cdots dy_n.$$

Then, by induction,

$$u_r(x_1, \cdots, x_n) = \sum_{m=0}^{r} (x_1x_2\cdots x_n)^m/(m!)^n.$$

Clearly $u(x_1, \cdots, x_n)=\lim u_r(x_1, \cdots, x_n)$ exists uniformly in any bounded region of n-space, and satisfies the integral equation. Thus a solution is

$$u(x_1, \cdots, x_n) = \sum_{m=0}^{\infty} (x_1x_2\cdots x_n)^m/(m!)^n.$$

Now, if u and v are two bounded measurable solutions and $w=u-v$, then

$$w(x_1, \cdots, x_n) = \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} w(y_1, \cdots, y_n)dy_1\cdots dy_n.$$

Hence, if M is a bound for $|w|$, then by induction

$$|w(x_1, \cdots, x_n)| \leq M|x_1x_2\cdots x_n|^m/(m!)^n \to 0$$

as $m\to\infty$, and u is unique.

Also solved by P. G. Engstrom, N. J. Fine, George Glauberman, Immanuel Marx, Chih-yi Wang, Robert Weinstock, and the proposers.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Theory of Storage. By P. A. P. Moran. Wiley, New York, 1960. 111 pp. $2.50.

The author, a pioneer in the application of probability theory to problems of water storage in dams, presents a concise and lucid account of the present status of the subject. In the first two chapters the mathematical preliminaries (including Markov processes) and inventory problems are briefly summarized. The core of the book is in the remaining four chapters: Dams—Discrete Time; Dams—Continuous Time; Monte Carlo and Other Statistical Methods; The Programming of Storage Systems. A bibliography consisting chiefly of post-1950 items bears witness to the novelty of the topic. This book can be recommended not only to the designer concerned with problems of dam construction but to any one interested in applications of recently developed techniques in probability theory.

H. KAUFMAN
McGill University

Classical Mathematics, A Concise History of the Classical Era in Mathematics. By Joseph Ehrenfried Hofmann. Philosophical Library, New York, 1959. 159 pp. $4.75.

This is a translation of Parts II and III of the author's *Geschichte der Mathematik* that appeared in Sammlung Göschen, Nos. 875 and 882. (Part I of this *Geschichte* had been translated earlier under the title *History of Mathematics.*) It presents a compact review of concepts and techniques that were discovered or developed during the periods known as the High and Late Baroque and the Age of Enlightenment. (*ca.* 1625–*ca.* 1790). The main characters in the grand drama of the discovery (or should it be "invention," "Erfindung" in the German original?) of the calculus and the development of power series are illumined succinctly and eruditely. Considerable space is given to details of the uncivil Newton-Leibniz War, Hofmann stating categorically that "the discovery of the calculus is exclusively to the credit of G. W. Leibniz." The book is clearly not for the novice. The initiate, however, will find many an interesting item in this concise history.

It is to be greatly regretted that the highly valuable bibliographies (covering over 80 pp.) have been curtailed in the translation. The price differential, too, is noteworthy; the German three booklets cost D.M. 7.20 (*ca.* $2.00); the English two volumes—$9.50.

PINCUS SCHUB
University of Pennsylvania

Complex Variables and Applications. By R. V. Churchill. McGraw-Hill, New York, 1960. $6.75.

The second edition of R. V. Churchill's *Complex Variables and Applications* preserves the virtues of the earlier edition. It is still the most suitable text for a one-semester course in complex variables, primarily intended for engineers and physicists, or for any other group which aims to reach the level of applications as rapidly as possible and spend only as much time on theory as is necessary to obtain a sound foundation for subsequent applications. Like most mathematicians, when the reviewer is trying to decide the merit of a complex variables book, his reaction is to study the section on integration. From this point of view the new edition seems to be a substantial improvement. There is a wide selection of applications in this book. In addition to the applications discussed in the first edition, we now have a chapter dealing with integral formulas of the Poisson type.

F. Haas
Wayne State University

Vector Analysis with Applications to Geometry and Physics. By Manuel Schwartz, Simon Green, and W. A. Rutledge. Harper, New York, 1960. xii+556 pp. $7.50.

The book might better be entitled "Topics from Geometry and Physics Done with Vector Analysis." Indeed, the authors regard vector analysis as a language of physics and attempt to train the reader to think in this language. This is done by interspersing applications with mathematical development. A large number of illustrative examples and exercises are supplied. Any reader brave enough to do the work called for will certainly emerge with considerable facility in the uses of vector analysis, and will learn a lot of geometry and physics along the way. It is also valuable as a reference work.

The chapter headings are worth quoting to illustrate the spirit of the book. They are: Vector algebra, Statics, Differentiation of vectors, Kinematics, Vector integral calculus, Dynamics, General coordinates, Differential geometry, Harmonic functions, Electrostatics, Magnetism and electrodynamics, Linear vector functions. An index is included.

Many difficult mathematical concepts are treated only cursorily, but a competent instructor can elaborate on them. The degree of mathematical rigor is quite variable. In many places theorems are stated precisely and proofs or references to same are supplied. In other places, even a precise statement of theorems is lacking. The reader must make additional differentiability assumptions to do many of the exercises. In short, sometimes it reads like a mathematics book and sometimes like a run-of-the-mill physics text. Despite this, the level of mathematical precision is, over-all, higher than that of most physics books on vector analysis.

Melvin Henriksen
Wayne State University

A Survey of Basic Mathematics. By H. G. Apostle. Little, Brown, Boston, 1960. xv+464 pp. $6.00.

Professor Apostle's book gives a year's terminal course for liberal arts students, though a semester course could be selected from it. It seems to the reviewer that so little is presupposed, only a bit more than elementary arithmetical operations, that in algebra and plane geometry there is too much review for many college students. Topics are chosen from logic, arithmetic, algebra, geometry, statistics, plane trigonometry, vector analysis, plane and solid analytic geometry and calculus. Brief appendices deal very well with equivalent systems, the number of primes and cardinal numbers. There are tables, as well as answers to odd-numbered problems.

Many of the exercises are well chosen and decidedly thought-provoking. The author is very careful about insisting on "checks" for problems, but not so careful about stating that a factor like $x-a$ may be taken from numerator and denominator only if nonzero, though he does give a good comment on this on page 124. The reviewer likes especially Professor Apostle's discussion of meaningless answers to algebraic problems, definition of cylinder with circular cylinder as special case, chapter on non-Euclidean geometries, presentation of probability and functions, choice of material on statistics, particularly comments on high coefficient of correlation vs. causation, discussion of vectors, good form for illustrative problems on progressions and amusing mnemonic devices for letters giving elements of *A. P.* and *G. P.*, applications of conics, manner of introducing calculus and habit of defining new concepts in terms of old ones.

The part of the book on analytic geometry and calculus calls them "higher mathematics." These topics seem to the reviewer *elementary* college mathematics, and they are sometimes taught in preparatory school. In trigonometry why spend so much time on solving triangles, when many present books are turning away from that topic? Careful correction of slight errors should be done for a second edition.

MARION E. STARK
Wellesley College

Théorie des graphes et ses applications. By Claude Berge. Dunod, Paris, 1958. viii+277 pp.

The tentacles of the theory of graphs steadily grow more numerous and penetrate more deeply into many phases of mathematics. The jargon of the theory includes such words and phrases as: arc, edge, path, chain, Hamiltonian circuit, tree, node, function of Grundy, latin square, incidence matrix, totally unimodular matrix, chromatic class, cyclomatic number, semi-factor, capacity, coupling, network. Applications of the theory spread over not only geometry, topology, and algebra, but also games, economics, military affairs, social structures, physics, statistics, communications, dynamic programming, operations research. Many famous questions which have helped to motivate the develop-

ment of the broad theory of graphs are seemingly unrelated. To illustrate their diversity, we mention: the problem of the eight queens on the chessboard; the promenade of the fifteen school girls; generalizations of Nim; the boat ferrying the wolf, goat, and cabbage across the river one at a time; the transportation problem; the excursion problem (where m families with $r_1, \cdots, r_m$ members respectively travel in n vehicles with $s_1, \cdots, s_n$ available seats respectively, and no family has two of its members in the same conveyance); the assignment of personnel; the path of a knight over every square on the chessboard; the design of a tournament so that the results will be "fair"; the traveling salesman problem; the seven bridges of Koenigsberg; the longest circular sequence composed of zeroes and ones with no repeated portion of k consecutive digits; the number of days in a conference where eleven ministers sit at a round table but no two ministers sit beside each other more than one day; the most effective bombardment of communication channels; the noncrossing of supply lines for the three public utilities to the three cities; the dissection of a rectangular region into pairwise incongruent square regions; the four-color problem. Of the several appendices, one lists fourteen questions still awaiting solution. All in all, in this book we have an up-to-date exposition, by one of the developers himself, of an intriguing theory capable of handling a fascinating potpourri of situations.

R. A. Good
University of Maryland and University of Oklahoma

Advances in Applied Mechanics, Vol. 6. Edited by H. L. Dryden et al. Academic Press, New York, 1960. x+294 pp. $9.00.

The founding editors (Von Mises and Von Kármán) of this series of volumes have explained clearly what the reader may expect to find here: "the principal aim of the *Advances* is to give surveys of the present state of research work in various fields of applied mechanics" (vol. 2, 1951); the volumes are "intended for students, scholars and engineers who are familiar with the contents of textbooks and handbooks and who are unable to follow up the research papers currently published" (vol. 1, 1948).

All the surveys in the present volume relate to fluid dynamics. Two surveys study boundary layers, the first dealing with unsteady laminar layers and the second with the phenomena of ionization and dissociation which are manifested in flows with very high velocities. The third studies shock waves in ducts of varying cross-section and the fifth examines theoretical and experimental work on Kármán vortex streets since 1953; interest in this topic has revived since it is applicable to technical processes, e.g. fuel sprays.

The fourth article is a detailed study (119 pp.) of similarity and equivalence in compressible flow.

The book has an international flavor; two of the contributors are in British universities and three in German research institutes.

John McNamee
University of Alberta

Introduction to Matrices and Linear Transformations. By Daniel T. Finkbeiner, II. Freeman, San Francisco, 1960. vii+248 pp. $6.50.

After giving in Chapter 1 a rather thorough introduction into *Abstract Systems*, the author continues in Chapter 2 with *Vector Spaces* to lay the foundation for the treatment of the subject. The remaining chapters are: 3. *Linear Transformations*, 4. *Matrices*, 5. *Linear Equations and Determinants*, 6. *Equivalence Relations on Matrices*, 7. *Characteristics of a Matrix*, 8. *A Canonical Form for Linear Transformations*, 9. *Metric Concepts*, and 10. *Functions of Matrices*. To supplement the discussion of abstract systems the first appendix formulates the necessary *Algebraic Concepts*. The second appendix, *Combinatorial Equivalence*, gives a thorough discussion of the pivot operations on matrices.

Throughout the book the wealth of exercises enhances the value of this excellent book as a text for a course. Both the notation and the printing format are very excellent. The book is extremely free from errors and is written in a very readable style.

ALBERT NEWHOUSE
University of Houston

Fonctions Hypergéométriques Confluentes. By F. G. Tricomi. (Mémorial des Sciences Mathématiques, No. 140) Gauthier-Villars, Paris, 1960. 86 pp. 20 NF (U. S. $4.27).

In this abridged version of his monograph *Funzioni Ipergeometriche Confluenti* (Cremonese, Rome, 1954; reviewed in Bull. Amer. Math. Soc., vol. 61, 1955, pp. 456–460), the author surveys the more important properties of confluent hypergeometric functions and illustrates through a few well-chosen examples the widespread applicability of these functions. By means of skillful organization, he is able to include all the dominant features of the parent Italian volume and to bind them together into a cohesive unit. For detailed proofs and for especially involved formulae, however, the reader is often referred to the parent volume or to other easily accessible literature. Not written to be an introduction to the subject (the author has already written an excellent one), nor at the other extreme to be a handbook (an excellent one by Herbert Buchholz is available), this little book very effectively summarizes the current state of knowledge of the subject.

C. A. SWANSON
University of British Columbia

Confluent Hypergeometric Functions. By L. J. Slater. Cambridge University Press, London-New York, 1960. ix+247 pp. $12.50.

This superbly produced volume contains in its first part the main properties of Kummer's and Whittaker's confluent hypergeometric functions, and in its second part the most extensive numerical tables of Kummer's function ${}_1F_1[a;b;x]$ which have yet been published. It assumes particular importance as a work of

reference because of the multitude of problems occurring in mathematics and physics that can be solved in terms of confluent hypergeometric functions. This is the first book in English to devote more than a chapter or two to the subject (although four books in Italian, German, and French have appeared since 1952; see this MONTHLY, present issue, for another review). It is written rather concisely, and contains a wealth of formulae and diagrams. About one half of the book comprises tables, computed by the author on EDSAC I in the Cambridge University Mathematical Laboratory.

An interesting feature is a very detailed chapter on asymptotic expansions. After producing various elementary results, the author proceeds to derive asymptotic expansions when both a and x in Kummer's functions become large together by appealing to some recent theorems of F. W. J. Olver. The same type of results were obtained in 1957 by A. Erdélyi and the reviewer (Memoirs Amer. Math. Soc., No. 25). The author could have increased the practical value of the asymptotic formulae by including statements of them in the important Coulomb wave function notation.

Finally, the extreme usefulness of this book as a permanent reference volume is accentuated by the inclusion of a good general index as well as symbolic index, and by the excellent printing and fine quality of the materials used.

C. A. SWANSON
University of British Columbia

Elementary Statistics. By Sidney F. Mack. Holt, Rinehart and Winston, New York, 1960. ix+198 pp. $4.50.

This text is written for the standard one-semester course in elementary statistics given in a mathematics department but largely to nonmathematics majors. As such, the contents are quite standard—summarization of data, fundamentals of probability and probability distributions, binomial and normal distributions, sampling distributions, confidence limits, testing hypotheses, including the use of Student's distribution and chi-square, and linear correlation and regression. An interesting addition is that of a refresher chapter going as far back as the arithmetic of fractions, which may be a useful addition for the type of audience that such courses attract. Also noteworthy is the brevity of the chapter on descriptive statistics; the obsolete concepts that frequently clutter up such chapters are here omitted. However, this reviewer deplores the obsolete definition of the variance with divisor n rather than $n-1$.

The probability treatment is based on the relative frequency definition of probability; several simple theorems on relative frequencies are proved "rigorously." There is also a sketchy section on empirical probability. The instructor may have a hard time with this and with the brief casual transition from discrete to (normal) continuous probabilities. Presumably he will have to take out some time to amplify the statement made on page 40, "we therefore refer to an event whose probability is 0 as an impossible event."

General definitions are frequently omitted, *e.g.*, population, confidence limits —though these concepts are illustrated or defined for special cases. Also point estimation is omitted and as a result it is necessary to introduce the least squares regression line simply as "the 'nearest' line to all the points." The careful setting out of the theorems is commendable even though most of these must be given without proof in a text at this level. However, it is questionable whether statistical rules of thumb should be glorified as theorems. Several of these were noted (Th. 6.3, 7.2, 9.1). The topics of confidence intervals and hypothesis testing appear to be well done with a modern (pre-decision theory) point of view within the framework established. In particular, the interpretations are carefully and correctly worded.

The reviewer regrets the absence of any approach to abstractions or generality but recognizes that it is debatable how much of this can be done with the students for whom this text is designed. The teacher who wishes to convey some statistical ideas to mathematically weak students, and who does not wish any abstract concepts, should give this book serious consideration among its numerous competitors.

DOUGLAS G. CHAPMAN
University of Washington

BRIEF MENTION

Unvollständigkeit und Unentscheidbarkeit. By Wolfgang Stegmüller. Springer-Verlag, Wien, Germany, 1959. 114 pp. $4.70.

This discussion, in German, of the results of Gödel, Church, Kleene, Rosser etc. of basic logic should be called to the attention of logicians among our readers. Presumably, it will be reviewed more extensively in the Journal of Symbolic Logic.

Nonlinear Problems in Random Theory. By Norbert Wiener. Wiley, New York, 1958. ix+131 pp. $4.50.

This book, which covers an amazing variety of topics, will be reviewed more fully in the *Bulletin* of the American Mathematical Society.

Ramification Theoretic Methods in Algebraic Geometry. By Shreeram Abhyankar. Princeton University Press, Princeton, N. J., 1959. vi+96 pp. $2.75.

This, too, will be reviewed in the *Bulletin* of the American Mathematical Society.

Automatic Data-Processing Systems. By Robert H. Gregory and Richard L. Van Horn. Wadsworth, San Francisco, 1960. xii+705 pp. $8.75.

Not a mathematics book, but a management-oriented book on data processing using high speed computers.

Instruction in Arithmetic. National Council of Teachers of Mathematics 25th Yearbook. NCTM, Washington, D. C., 1960. viii+366 pp. $4.50. To members of the Council $3.50.

Brief Course in Analytics (3rd ed.). By M. A. Hill, Jr. and J. B. Linker. Holt, New York, 1960. viii+232 pp. $3.90.

Analytic Geometry (3rd ed.). By John W. Cell. Wiley, New York, 1960. xii+330 pp. $4.95.

Differential Equations (2nd ed.). By Alfred L. Nelson, Karl W. Folley and Max Coral. Heath and Co., Boston, Mass., 1960. x+308 pp. $5.25.

Elementary Differential Equations (5th ed.). By Lyman M. Kells. McGraw-Hill, New York, 1960. x+318 pp. $6.25.

Calculus and Analytic Geometry (3rd ed.). By George B. Thomas, Jr. Addison-Wesley, Reading, Mass., 1960. xii+1010 pp. $10.75.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. A. Rosenbaum, Wesleyan University, represented the Association at the Silver Convocation honoring Dr. A. N. Jorgensen, President of the University of Connecticut, on November 12, 1960.

California Institute of Technology: Professor H. Wielandt, University of Tubingen, Tubingen, Germany, has been appointed Visiting Professor for the academic year 1960–61; Dr. G. D. Chakerian, University of California, Berkeley, has been appointed Instructor; Dr. E. C. Dade, Princeton University, has been appointed Bateman Research Fellow in Mathematics; Dr. R. D. Ryan has been appointed Research Fellow in Mathematics; Assistant Professors W. A. J. Luxemburg and C. H. Wilcox have been promoted to Associate Professors; Dr. R. E. Block has been promoted to Assistant Professor.

Carleton College: Associate Professor John Dyer-Bennet, Purdue University, has been appointed Associate Professor; Dr. N. W. Johnson, University of Toronto, and Mr. Joseph Woolfson, Brooklyn College, have been appointed Instructors; Mr. P. S. Jorgensen is on leave at the University of Copenhagen, Denmark, for the academic year 1960–61; Assistant Professor F. L. Wolf has been promoted to Associate Professor and is on leave at the University of California, Berkeley, for the year 1960–61.

Eastern Michigan University: Mr. D. D. Heikkinen and Mr. G. D. Anderson have been appointed Assistant Professors; Mr. W. M. Fitzgerald has been appointed Instructor.

Indiana University: Professor P. R. Masani, Brown University, has been appointed Professor; Professor L. C. Young, University of Wisconsin, has been appointed Visiting Professor; Dr. R. J. Troyer has been appointed Instructor.

Montana State College: Dr. C. P. Quesenberry, Virginia Polytechnic Institute, and Dr. H. R. Fischer, Swiss National Foundation Research Fellow, have been appointed Assistant Professors; Mr. J. L. Simpson has been promoted to Assistant Professor; Assistant Professor C. J. Mode has been promoted to Associate Professor.

Trenton State College: Mr. L. B. Sklar, Rutgers University, has been appointed Assistant Professor; Assistant Professor John McIlroy has been promoted to Associate Professor.

University of Arizona: Visiting Professor Berthold Schweizer, University of California, Los Angeles, Dr. Paul Slepian, Hughes Research Laboratories, Culver City, California, and Assistant Professor H. M. Lieberstein, University of Wisconsin, have been appointed Associate Professors.

University of Colorado: Dr. Wolfgang Schmidt, University of Vienna, Vienna, Austria, has been appointed Assistant Professor; Dr. Marguerite Dunton has been appointed Acting Assistant Professor; Assistant Professors Arne Magnus, B. C. Meyer, and Robert McKelvey have been promoted to Associate Professors; Professor Sarvadaman Chowla is on leave as Visiting Professor at the University of Notre Dame; Professor Robert McKelvey is on leave at the University of California, Los Angeles.

University of Georgia: Assistant Professor L. W. Anderson, University of Oregon, and Dr. R. W. Heath, Woman's College of the University of North Carolina, have been appointed Assistant Professors.

University of Minnesota: Dr. H. F. Weinberger, University of Maryland, has been appointed Associate Professor; Dr. S. A. Gal, Yale University, has been appointed Visiting Associate Professor; Assistant Professor Walter Littman, University of Wisconsin, and Dr. E. R. Rodemich, Massachusetts Institute of Technology, have been appointed Assistant Professors; Associate Professors Eugenio Calabi, Lawrence Markus, J. C. C. Nitsche and J. B. Serrin, have been promoted to Professors; Drs. W. A. Harris, Jr., and Richard Juberg have been promoted to Assistant Professors.

University of Utah: Assistant Professor R. C. Bzoch, University of Minnesota, has been appointed Assistant Professor; Associate Professor Edward Nelson, University of North Dakota, has been appointed Assistant Professor; Assistant Professors W. J. Coles, E. A. Davis, E. E. Kohlbecker, and D. V. V. Wend, have been promoted to Associate Professors.

University of Wisconsin: Assistant Professor Fred Brauer, University of British Columbia, Drs. M. I. Knopp and F. A. Raymond, Institute for Advanced Study, and Dr. J. E. Ohm, University of California, have been appointed Assistant Professors; Drs. A. N. Feldzamen, University of Chicago, P. E. Miles, Yale University, and S. H. Coleman, University of Virginia, have been appointed Instructors.

University of Wisconsin—Mathematics Research Center: Drs. Creighton Buck, Millard Johnson, Walter Rudin, Hans Schneider, and Kennan Smith, University of Wisconsin, have joined the staff as part-time members. Drs. Donald Greenspan and Johan Kemperman, Purdue University, Robert Moore, recently returned from study in Germany, Sam Saunders, Boeing Airplane Company, Seattle, Washington, Jose Gonzalez-Fernandez, Institute of Mathematical Sciences, J. Schroder, Hamburg, Germany, Peter Werner, Aachen, Germany, Edwardo Zarantonello, Argentina, and Professor O. Bjorgum, Norway, have joined the staff for this year.

University of Wisconsin—Milwaukee: Drs. F. W. Caroll, University of Amsterdam, Amsterdam, Netherlands, H. C. Howard, University of Wisconsin, Madison, Morris Katz, and Togo Nishiura, have been appointed Assistant Professors; Mr. R. H. Black and Mr. J. E. McAdam have been appointed Instructors.

University of Wyoming: Professor J. R. Hanna, University of Wichita, has been appointed Associate Professor; Mr. Kenneth Batker, University of Colorado, and Mr. Richard Finley have been appointed Instructors.

Washington University: Professor K. A. Hirsch, Queen Mary College, University of London, London, England, has been appointed Visiting Professor; Assistant Professor R. H. McDowell, Rutgers University, has been appointed Assistant Professor; Assistant Professor F. J. Schnitzer, Wayne State University, has been appointed Research Associate and Visiting Assistant Professor; Associate Professor Guido Weiss, DePaul University, has been appointed Associate Professor and granted leave to accept a National Science Foundation Fellowship for research in Europe for the academic year 1960–61;

Associate Professor Allen Devinatz is on leave for one year to accept a Senior National Science Foundation Postdoctoral Fellowship to undertake research at the Institute for Advanced Study; Professor I. I. Hirschman, Jr., is on leave for one year to do research in Zurich, Switzerland, under an Air Force contract.

Wesleyan University: Drs. S. L. Salas, Yale University, and H. J. Arnold, University of Western Ontario, London, Ontario, Canada, have been appointed Assistant Professors; Dr. G. P. Johnson, Standard Oil Company of California, San Francisco, California, has been appointed Associate Professor; Associate Professor Hing Tong has been promoted to Professor.

Wheaton College (*Massachusetts*): Associate Professor Barbara J. Beechler, Wilson College, has been appointed Associate Professor; Associate Professor Anne F. O'Neill has been promoted to Professor; Professor C. A. Garabedian retired June, 1960, with the title Professor Emeritus.

Miss Martha Anderson, Wisconsin State College, has been appointed Assistant Instructor at the University of Kansas.

Mr. J. A. Brown, Montana State College, has been appointed Research Worker at Johns Hopkins University Operations Research.

Miss Joyce C. M. Cimelus, Butler University, has been appointed Instructor at the University of Bridgeport.

Mr. B. F. Edwards, Jr., Chance Vought Aircraft, Dallas, Texas, has been appointed Assistant Professor at Stephen F. Austin State College.

Professor Tomlinson Fort, University of Georgia, has been appointed Professor at the University of Miami.

Research Professor B. E. Gatewood, Air Force Institute of Technology, has been appointed Professor at The Ohio State University.

Assistant Professor Artur Grigori, St. Bonaventure University, has been appointed Assistant Professor at the University of Redlands.

Dr. John Gurland, Iowa State University, has been appointed Professor at the Mathematics Research Center, U. S. Army, University of Wisconsin.

Assistant Professor B. W. Helton, University of Utah, has been appointed Associate Professor at Southwest State College.

Mr. R. R. Korfhage, University of Michigan, has been appointed Assistant Professor at North Carolina State College.

Mr. Knox Millsaps, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, has accepted the position of Executive Director at the Air Force Office of Scientific Research, Washington, D. C.

Mr. J. S. Moore, Jr., Florida Christian College, has been appointed Director of the Division of Terminal Electronics. He continues his position as Head of the Division of Physics and Mathematics.

Professor Seymour Parter, Indiana University, has been appointed Professor at Cornell University.

Mr. R. E. Rundus, Park College, has been appointed Assistant Professor at Northern Illinois University.

Mr. R. F. Rutschow, Pennsylvania State University, has been appointed Instructor at the Virginia Military Institute.

Mr. B. F. Ryder, Balboa High School, Balboa, Canal Zone, has been appointed Teacher at Weymouth High School, East Weymouth, Massachusetts.

Sister Maris Stella Schrot, University of Notre Dame, has been appointed Teacher at St. Bernadine High School, San Bernardino, California.

Dr. H. R. Stevens, Duke University, has received a Fulbright Award for study at the University of Hamburg, Hamburg, Germany, during the year 1960–61.

Mr. J. E. Strout, University of Illinois, has been appointed Instructor at Indiana State Teachers College.

Dr. L. H. Turner, Space Technology Laboratories, Los Angeles, California, has been appointed Assistant Professor at the University of Minnesota.

Assistant Professor M. J. Walsh, University of Wyoming, has accepted a position with the Martin Company, San Diego, California.

Mr. R. J. Winterbottom, III, Ursinus College, has accepted a position as Service Engineer with the Spinco Division of Beckman Instruments, Palo Alto, California.

Mr. Walter Zayachkowski, University of Alberta, Edmonton, Alberta, Canada, has been appointed Lecturer at Essex College, Windsor, Ontario, Canada.

Professor Alfred Errera, Uccle, Belgium, died September 18, 1960. He was a member of the Association for twenty-one years.

Professor B. M. Ingersoll, Arizona State University, died October 3, 1960. He was a member of the Association for seven years.

Professor G. B. Lang, University of Florida, died July 21, 1960. He was a member of the Association for twenty-five years.

Professor G. E. Moore, Eastern Michigan University, died during the summer, 1960. He was a member of the Association for thirty-seven years.

Professor W. L. Fields, Hampton Institute, died May 29, 1960. He was a member of the Association for nineteen years.

Sister Mary Clementia, S.S.F., St. Mary's Academy, New Orleans, Louisiana, died March 7, 1960.

GRADUATE LABORATORY DEVELOPMENT PROGRAM

The National Science Foundation announces that March 1, 1961, is the next closing date for receipt of proposals in the Graduate Laboratory Development Program. Proposals received after that date will be reviewed following the next closing date, September 1, 1961. This program requires at least 50% participation by the institution with funds derived from non-Federal sources.

The purpose of the grants is to aid institutions of higher education in modernizing, renovating, or expanding graduate-level basic research laboratories used by staff members and graduate students. Only those departments having an on-going graduate training program leading to the doctoral degree in science at the time of the proposal submission are eligible at present.

Proposals, as well as requests for additional information, should be addressed to: Office of Institutional Programs, National Science Foundation, Washington 25, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FILMS BY MCSHANE AND HENKIN

The Mathematical Association of America announces the availability of two mathematical films for classroom use.

For high school seniors and college freshmen and sophomores:

MATHEMATICAL INDUCTION by Professor Leon Henkin, University of California, Berkeley. (Two reels, each 30 minutes.)

For college juniors and seniors and beginning graduate students:

THE THEORY OF LIMITS by Professor E. J. McShane, University of Virginia. (Four reels, averaging 15 minutes each.)

These films are accompanied by a booklet containing the complete written script, supplementary material for class discussion, and problems for the students. It is suggested that some class discussion of the material presented in the film follow each reel.

These films are being distributed by the Association on a loan basis without charge except for postage. Requests for films should be directed to: Committee on Production of Films, Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Anyone interested in purchasing these films should also write to the above address.

A STUDY OF THE DESIGN OF FACILITIES FOR MATHEMATICS

As its first project, the Conference Board* will conduct a study of the design of buildings and facilities for mathematics; Educational Facilities Laboratories, established by the Ford Foundation in New York City, has agreed to provide the necessary funds.

There are many reasons why it is desirable to make a study of the design of facilities for the mathematical sciences at the present time. In the first place, mathematics has been very poorly housed in the past. In the second place, enrollments are now expanding rapidly. Many colleges and universities have five times as many majors in mathematics as they had only four or five years ago, and the great increases in enrollments for the nation as a whole are still to come.

In the third place, a study of the design of mathematics facilities is appropriate because of the many changes that have taken place in the mathematical sciences. The project will undertake a study of the design of facilities to support the total activities of the mathematical sciences. These activities include research and instruction in pure mathematics, applied mathematics, and statistics; preparation of the manuscripts of research papers; preparation of the manuscripts of textbooks and expository manuscripts for instructional purposes; teacher training; instruction in the operation of desk calculators and electronic digital computers; and the operation of summer and academic year institutes. Modern facilities for the mathematical sciences must provide headquarters space; classrooms; seminar rooms; offices for the staff; library space; a statistics laboratory with desk calculators; a computation center for the electronic digital computer; facilities for the use of films, television, and other teaching aids; and a common room.

The construction of appropriately designed facilities for mathematics is important for a special reason at this time. There is a great shortage of mathematics teachers, and it is probable that this shortage will continue for many years. Under these conditions it is imperative that we make the teachers we do have more efficient than they have been in the past. Some universities are teaching elementary courses in sections of one to two hundred students; others would like to do so, but lecture rooms for classes of this size are not available. In many cases several staff members—even senior staff members—are crowded into one office. When one has a visitor, the others stop work. There are important universities that have never been able to provide one chair and desk per staff member. Better classrooms, better offices, and better facilities of all kinds will certainly make our mathematics staffs more efficient—will enable our mathematics teachers to teach more students and to teach them better.

In the study of the design of facilities for mathematics it will be kept in mind that mathematics is a peculiarly human and personal activity. Some academic subjects involve work with laboratory equipment or machines, but the typical mathematician sits

* This MONTHLY, vol. 67, 1960, pp. 1056–1057.

at a desk and works with pencil and paper. If he is to be productive of ideas, he must be in pleasant, comfortable, and congenial surroundings.

The project will begin by collecting information about good features of buildings already in existence, both in the United States and abroad. It will be necessary to hold a small conference at the beginning of the project to consider the nature of the activities to be conducted in a center for the mathematical sciences, and to explore the design of facilities. The end result of the project will be a report which can be published commercially. The staff will consist of a mathematician and a secretary, and architectural services will be provided by an architectural firm in Washington, D. C. The project, expected to be completed in one year, will begin as soon as the staff has been organized.

The project will include a study of mathematics facilities for high schools, and the final report will include a section or chapter on mathematics classrooms and other facilities for high schools. This part of the report will be published separately so that it will be easily available to high school administrators and others interested only in the high school field.

G. Baley Price

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

Allegheny Mountain, West Virginia University, Morgantown, May 6, 1961.

Illinois, University of Illinois, Urbana, May 12–13, 1961.

Indiana, Rose Polytechnic Institute, Terre Haute, May 6, 1961.

Iowa, Simpson College, Indianola, April 14, 1961.

Kansas, Ottawa University, April 15, 1961.

Kentucky, Western Kentucky State College, Bowling Green, Spring, 1961.

Louisiana-Mississippi, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

Maryland-District of Columbia-Virginia

Metropolitan New York

Michigan, Wayne State University, Detroit, March 25, 1961.

Minnesota

Missouri, University of Missouri, Columbia, April 22, 1961.

Nebraska, University of Nebraska, Lincoln, April 15, 1961.

New Jersey

Northeastern

Northern California

Ohio, Ohio Wesleyan University, Delaware, May 6, 1961.

Oklahoma

Pacific Northwest, University of Washington, Seattle, June 17, 1961.

Philadelphia, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

Rocky Mountain, University of Colorado, Boulder, April 28–29, 1961.

Southeastern, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

Southern California, University of California, Santa Barbara, March 11, 1961.

Southwestern, University of Arizona, Tucson, April, 1961.

Texas, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

Upper New York State, Harpur College, Binghamton, April 29, 1961.

Wisconsin, University of Wisconsin, Madison, May 13, 1961.

One of a series.

Why we eliminated the earth's magnetic field . . . almost

In an isolated laboratory in southwestern Ohio, GM Research scientists have reduced the earth's magnetic field to one ten-thousandth of its usual strength. This is about as weak as the interplanetary field detected by the Pioneer V solar satellite.

Why neutralize the earth's field? To perform with precision one of the more fundamental experiments in magnetism—measuring the Einstein-DeHaas effect. The measurement is simple in concept, experimentally difficult because of the tiny forces involved. It is made by suspending a ferromagnetic rod in a nearly field-free environment . . . magnetizing the rod . . . then measuring the effect (how much the rod rotates) when this known magnetization is reversed.

The beauty of the experiment is that the resulting values can be related directly to the motions of electrons in the rod. The values indicate the large portion of magnetization due to the *spin* of electrons . . . and the slight, but theoretically important, remaining portion due to *orbital* motion of electrons.

These measured values are helping scientists form a better understanding of the perplexing phenomenon—ferromagnetism. Currently being pursued in cooperation with the Charles F. Kettering Foundation, this long-standing project is one of the ventures in basic research of the General Motors Research Laboratories.

General Motors Research Laboratories
Warren, Michigan

System of Helmholtz coils used to neutralize earth's magnetic field.

A COMPREHENSIVE SELECTION

. . . NEW—COMING IN SPRING, 1961 . . .

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Zeev Nehari

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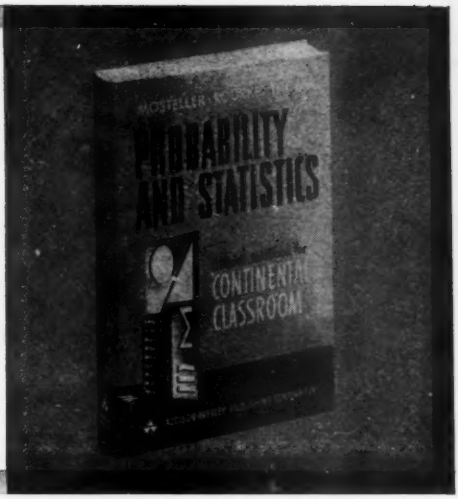

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PART II

A Contemporary Approach to Classical Geometry

WALTER PRENOWITZ

Number 9
of the
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MEMORIAL PAPERS

JANUARY 1961

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A CONTEMPORARY APPROACH TO CLASSICAL GEOMETRY

By

WALTER PRENOWITZ

Professor of Mathematics

Brooklyn College

The Ninth

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 68 Number 1

January, 1961

PREFACE

This paper presents an approach to classical geometry that attempts to be modern in spirit—broad in scope, conceptually oriented, synthetic in treatment in that it deals with the subject matter directly—but well grounded in geometric intuition. It aims to be abstract and rigorous but tries to avoid getting mired in the esoteric subtleties of the foundations of geometry as conventionally studied. The treatment is centered on a study of the join operation applied to two figures, for example, the construction of a pyramid or cone by joining a point to a set of points. The burden is put on the simplest case of this operation, namely that of joining two points to form a segment.

The treatment is an outgrowth of my study of classical geometries and their generalizations (projective geometries, ordered and partially ordered linear geometries, spherical geometries) in terms of an operation of joining two points to form a basic connective (line, segment, circular arc). It is similar in spirit to the lattice-theoretic approach to geometry of Birkhoff and of Menger: it is operation-centered rather than configuration-centered and deals with geometrical material (of arbitrary dimension) directly, not through the intervention of coordinate representation. It differs from their approaches in that the basic operation applies initially only to points, not higher dimensional linear spaces. This makes possible the framing of postulates that are directly verifiable in simple diagrams. Since the treatment is based on a single binary operation (joining) it bears closer analogies to group theory than to lattice theory. These become rather strong as the development progresses.

The paper assumes no *formal* knowledge of geometry and is essentially complete in itself but the reader will not find that mathematical maturity hampers his comprehension. To expedite development of the theory much material giving important examples and counterexamples has been placed in an appendix, which however should be considered an integral part of the paper, as it adds richness to the theory and should aid in its comprehension. The major part of the appendix will be understandable after Section 9 of the paper is finished (some of it earlier) and the reader is encouraged to examine it as he reads the paper.

I wish to express my gratitude to the many students and associates who read and criticized a preliminary version of a portion of the manuscript. In particular my warm thanks are extended to Meyer Jordan, Abraham Karrass and Donald Solitar for their devotion in reading and criticizing the manuscript and making suggestions for its improvement. Some of the material presented here was originally tried out in various forms in my classes, and I must express deep satisfaction with my students at Brooklyn College for their stimulation and appreciation of a teacher's efforts to rethink his subject.

W. P.

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I. THE CONCEPT OF JOIN SYSTEM AND ITS BASIC PROPERTIES

1. Introduction.* If Euclid returned to earth today he would find our basic approach to geometry and our way of conceiving it not very different from his own. I venture the guess that if Euclid were to examine the famous work of Hilbert on the foundations of geometry [5], he would have more trouble with the German than the geometry. It is true that Hilbert's treatment is more precise and more rigorous than Euclid's, but the basic spirit remains unchanged. The propositions still tend to be direct transliterations of raw visual data, and are kept in mind mainly through visual intuition. The proofs usually are pictorially motivated and remembered. There are no general concepts and methods which serve to unify the material, such as we find in modern axiomatic algebra or modern analysis. As a result, we frequently encounter special or degenerate cases, so that to give a complete proof of an important theorem often is a triumph of character rather than intellect.

In the Euclid-Hilbert treatment of geometry it is hard to find an *intrinsic* motivation for the choice of objects studied or the choice of postulates. For example why study a triangle as a broken line, rather than a triangular region? Why an angle, rather than the figure formed by any two rays? Why apart from naive visual experience assume that a line is separated into three parts by two of its points?

Let me make the point in a somewhat different way: The tremendous outpouring of geometrical knowledge in the 19th century involving projective geometry, non-Euclidean geometry and n-dimensional geometry has not affected our basic approach to classical geometry. The new subjects form a sort of historical addition to the edifice—it has not been rebuilt in the light of them. Higher geometry is not an outgrowth of basic geometry, but a more or less related subject, studied primarily by *analytic* methods. Is it any wonder that many students find their experience in geometry unsatisfying and lacking the elegance, rigor and conceptual unity of modern algebra and analysis? Is it any wonder that geometry seems uninviting to many students and teachers, and seems almost to be disappearing from the college curriculum?

Surely, by the middle of the 20th century, a serious attempt should be made to rethink basic geometry in the light of the great 19th century advances, in order to achieve at least a minimum of conceptual attractiveness. If we default or fail in this attempt, geometry may disappear as an autonomous branch of mathematics and be reduced to a graphical way of describing certain results in the algebra of n variables.

My object is to call this problem to your attention and to suggest a way of dealing with part of it.

Following Hilbert [5] we may take as the basic divisions of classical geom-

* This paper is an amplification of a lecture which was presented by invitation to the Mathematical Association of America at Rochester, N. Y., December 29, 1956.

etry the theories of incidence, of order, and of congruence. I am going to indicate a treatment of the theories of incidence and order which is completely independent of dimension, and so is applicable to geometrical systems of arbitrary dimension, finite or infinite. To achieve this, the Euclid-Hilbert visual-centered concepts, bound down as they are to the line and the plane, must be replaced by dimensionally neutral ideas. Such ideas are well known in geometry and have received recent attention in the algebra of linear inequalities, functional analysis, game theory and linear programming. I have in mind the notion *convex set* and its important specializations, *linear space* and *half-space*. Surely these should be central in a 20th century formulation of basic geometry.*

2. The operations of joining and extending. How can we organize the material around the idea of convex set? By definition, a set is convex if it contains the segment joining each pair of its points—that is, if it is closed under the operation of joining two points to form a segment. But this is precisely the basic operation in Euclid. My point of departure is very simple, it is just this: *To take the operation of joining two points to form a segment as fundamental, and to throw the burden of unifying our material on the consistent and relentless exploitation of this operation.*

Let me show you the concrete geometric basis for the procedure. We take the join of two distinct points a, b to be the open interval or segment ab joining a and b, considered as a set of points. Now if our operation is not to be artificially restricted, it is essential that it apply equally well whether $a \neq b$ or $a = b$. Thus we must clarify the idea of the join of a and a—this we naturally take to consist just of a.

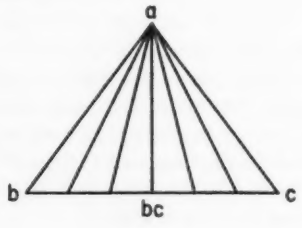


FIG. 1

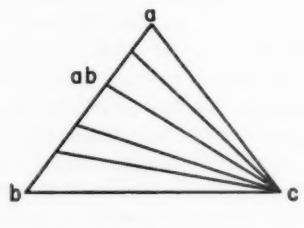


FIG. 2

Can we form the join of more than two points? For example consider three points a, b, c, say the vertices of a triangle. To get their join we first join b to c getting segment bc. Then we join a to each point of bc and aggregate all the joins formed. Clearly this yields the interior of $\triangle abc$ (Fig. 1). You may observe it is just as reasonable to "obtain" the join of a, b, c by a different process: Join each point of ab to c and aggregate the joins formed. Clearly this also yields the in-

* For an axiomatization of ordered linear geometry based on the notion of convex set see W. Prenowitz, Total lattices of convex sets and linear spaces, Ann. of Math., vol. 49, 1948, pp. 659–688.

terior of $\triangle abc$ (Fig. 2). If we describe the respective results of the two processes as the join of a to bc and the join of ab to c, we may assert the equality of these results as a sort of associative law for the operation "join."

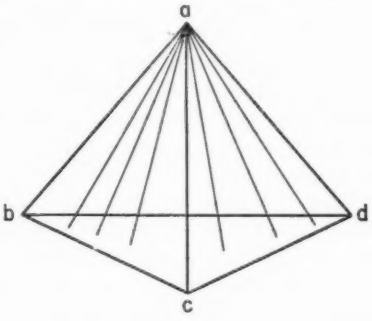


FIG. 3

Similarly let a, b, c, d be the vertices of a tetrahedron. To get their join we first form the join of b, c, d. This of course is the interior of $\triangle bcd$. Then we join a to each point of this and aggregate the joins formed, thus obtaining the interior of tetrahedron $abcd$ (Fig. 3).

Observe that we can not merely construct the join of two points, or of a point and a set, but also the join of two sets of points. For example take segments ab, cd in the last illustration. We get their join by forming the set union of all joins obtained by joining a point of ab and a point of cd. This also is seen to be the interior of tetrahedron $abcd$.

We now formalize the preceding discussion. In a Euclidean geometry we define the *join* of *points* a, $b (a \neq b)$ to be the segment whose endpoints are a, b—that is, the set of points lying strictly between a and b and so *excluding* a and b. The *join* of point a and itself we define to consist just of a. It is essential to extend our operation from points to sets of points, in order to be able to apply it to several terms and to general geometric figures. Thus we define the *join* of the *sets of points* A, B to be the set formed by joining each point of A to each point of B and aggregating all joins formed in this way. If A consists of a single point a we call the result simply the join of a and B; similarly if B reduces to point b, we refer to the join of A and b.

We now state the fundamental properties of the operation join in Euclidean geometry. They are easily verified pictorially and can be deduced formally from the order postulates of Euclidean geometry.* Letters a, b, c refer to arbitrary points.

(A) (CLOSURE LAW). *The join of a and b is a nonempty set of points.*

* For a formulation of these postulates see Veblen ([13], [14]), Forder ([4], pp. 42–48) or Coxeter ([2], pp. 161–162).

(B) (COMMUTATIVE LAW). *The join of a and b is identical to the join of b and a.*

(C) (ASSOCIATIVE LAW). *The join of a with the join of b and c is identical to the join with c of the join of a and b.*

(D) (IDEMPOTENT LAW). *The join of a and a consists of a.*

Although the basic operation in Euclid is to join points to form segments, it is not the only important operation. Next in importance we must consider the operation of extending or prolonging segments to form half-lines or rays. We can easily characterize *extend* in terms of *join*. Formally we define the *extension of*

FIG. 4

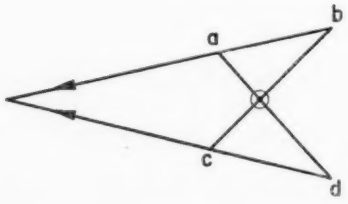


FIG. 5

a from b to be the set of points x whose join to b contains a. If $a \neq b$ this is the *open* half-line or ray which emanates from a and is directed away from b (Fig. 4). If $a=b$ (and our definition is significant for this case also) it consists solely of a. The basic properties of extension are:

(E) (CLOSURE LAW). *The extension of a from b is a nonempty set of points.*

(F) (TRANSPOSITION LAW). *Let the extension of a from b meet the extension of c from d. Then the join of a and d meets the join of b and c* (Fig. 5).

(G) (IDEMPOTENT LAW). *The extension of a from a consists of a.*

We conclude with some observations on properties (A), · · · , (G). It cannot be emphasized too strongly that these properties—unlike most current formulations of geometrical principles—are completely general and hold for all degenerate or "limiting" cases. For example (C), the associative law for join, holds if a, b, c are distinct and collinear or if $a=b$ or even if $a=b=c$. Similarly for (F), the transposition principle.

The transposition principle is very important since it gives a formal relation between join, the basic operation, and extend, the derived one. It is a generalization in "join language" of a triangle postulate used by Peano which may be

stated: *Segments which join two vertices of a triangle to respective points of their opposite sides intersect.*

Property (G), the idempotent law for extension, is a direct consequence of the fact that a segment does not contain its endpoints. For certainly a is a point of the extension of a from a—since the join of a to a is a by property (D). Suppose a point x distinct from a is contained in the extension of a from a. Then by definition, the join of x to a contains a. That is, the segment whose endpoints are x and a contains a. This is impossible. Hence the extension of a from a contains only point a.

It may seem preferable to define the join of two distinct points in Euclidean geometry to be a closed interval rather than an (open) segment. This is not justified for several reasons. First, if this were done, we would lose the formal simplicity of the idempotent law for extension—in fact the extension of a from a would contain all points. Second, it is formally simpler, as a set theoretic operation, to adjoin its endpoints to a segment to form a closed interval, rather than the reverse—it is easier to study unions, rather than differences, of sets. Finally, open geometric figures (for example, segments, rays, triangle interiors) are in a sense simpler to study than closed ones, since they are "homogeneous"—all points are "interior" points.

Our properties (A), $\cdots$, (G) are too weak to characterize Euclidean geometry. Much has been omitted. We have left out (1) a parallel postulate; (2) any reference to congruence; (3) the basic incidence property, "two distinct points belong to a unique line." Moreover (A), $\cdots$, (G) do not imply that the points on a line* are "fully ordered." In join language this may be stated: *If three distinct points colline, one of them is in the join of the other two.* Finally note that we have not included any dimensionality restriction. Thus (A), $\cdots$, (G) hold for Euclidean geometries of arbitrary dimension.

3. Abstract geometry based on join. It seems desirable to formulate an abstract theory of geometry based on properties (A), $\cdots$, (G) of the join operation. This will have several advantages: (1) It will enhance rigor of argument by removing any possible pictorial basis for verification of theorems. It will make our treatment logically independent of, though still applicable to, school geometry. (2) It will broaden greatly the scope of interpretation of the theory, since (A), $\cdots$, (G) are not a full complement of postulates for Euclidean geometry. (3) It will afford a deeper insight into the relations of algebra and geometry than is otherwise obtainable, since the geometric theory, like algebra, will be operationally based.

Before beginning the formal treatment we explain the notational conventions which we shall adopt. As is customary, we use $\cup$, $\cap$ for union and intersection of two sets, and $\bigcup$ for union of an arbitrary system of sets. $\varnothing$ denotes the empty or void set. The finite set whose elements (not necessarily distinct) are $a_1, \cdots, a_n$ is denoted $(a_1, \cdots, a_n)$.

* The formal sense in which we use the term "line" is stated in Section 9 below.

It will be very helpful to make certain other less-conventional simplifications of notation. In our theory, unlike algebra, we must be concerned with sets from the beginning, since the join of two elements will be a *set*. Thus we must study the set containment relations $\supset$, $\subset$ as well as element containment, $\in$. Furthermore, relations involving arbitrary sets will have valid degenerate cases in which the sets "reduce" to single elements. As a trivial example suppose $A \supset B$ and $B = (b)$, the set whose only element is b. In this case we write $A \supset b$ or $b \subset A$ instead of $A \supset (b)$. That is, we shall not formally distinguish between set (b) and element b. This permits us to dispense with the $\in$ relation altogether, and as we shall see yields other simplifications.

Furthermore, again unlike algebra, we shall constantly have to study the important geometric *relation*, intersection. That is, we say set A *intersects* or *meets* set B if $A \cap B \neq \varnothing$. This is awkward to deal with symbolically. Thus we adopt the simple form $A \approx B$ for A meets B. This turns out to be very useful. If $A = a$ and $B = b$ the relation $A \approx B$ reduces to the equality $a = b$. In general the relation $\approx$ will behave as a sort of weak equality relation—it is reflexive and symmetric but not transitive. Also it covers "element containment" relations—for example $A \supset b$, where b is an element, is equivalent to $A \approx (b)$ which we write simply $A \approx b$.

Now to proceed formally, consider an abstract system $(G, \cdot)$ in which G is a set of elements $a, b, c, \cdots$ and $\cdot$ is a 2-term operation which associates to each pair of elements a, b of G a uniquely determined set called the *product* or *join* of a and b, and denoted $a \cdot b$ or ab. Our system satisfies the following postulates.

(J1) (Closure Law). $\varnothing \neq ab \subset G$.

Thus ab is a uniquely determined nonvoid subset of G.

(J2) (Commutative Law). $ab = ba$.

In order to state the associative law and in general to extend our operation to several terms we must define the join AB of the subsets A, B of G. This we define to be the set of elements obtained by joining each element of A to each element of B and aggregating the results. Formally stated:

$$AB = \bigcup_{a \subset A, b \subset B} (ab).$$

In simple terms this means: $x \subset AB$ if and only if there exist a, b such that $x \subset ab$ and $a \subset A$, $b \subset B$. If A (or B) reduces to a single element a (or b) we write AB simply as aB (or Ab) in view of our agreement to "identify" a unit set with its element. Then we can state

(J3) (Associative Law). $(ab)c = a(bc)$.

Can our operation join be inverted? By analogy with ordinary algebra we might be tempted to say a/b is the unique solution x of $bx = a$. However bx is

not in general an element but a set of many elements—thus it is more natural to consider the relation $bx \supset a$. In this relation a is so to speak one of the values of bx, and there may be many x for which the relation holds. Thus we define a/b (read a divided by b or a over b) to be the set of all solutions x of the relation $bx \supset a$. We call a/b the *extension* of a from b.*

(J4) (CLOSURE LAW). $\varnothing \neq a/b \subset G$.

(J5) (TRANSPOSITION LAW). *If* $a/b \approx c/d$ *then* $ad \approx bc$.

(J6) (IDEMPOTENT LAWS). $aa = a = a/a$.

We call a system $(G, \cdot)$ which satisfies (J1), $\cdots$, (J6) a *join system* or a *join geometry*. Sometimes if there is no ambiguity in the specification of the operation $\cdot$, we simply refer to the join system G.

The application to Euclidean geometry is immediate. Interpret G to be the set of points of a Euclidean geometry and $a \cdot b$ to be the join of points a, b as defined in Section 2. Then a/b is the extension of a from b as defined there and (J1), $\cdots$, (J6) become properties (A), $\cdots$, (G). Consequently $(G, \cdot)$ is a model or interpretation of postulate system (J1), $\cdots$, (J6) or as we say a *join system*.

Observe in this model the intuitive-geometric relation between join and extension: segment $a \cdot b$ is described by moving from a toward b, ray a/b by moving from a directly away from b (Fig. 6). Moreover the important decomposition of

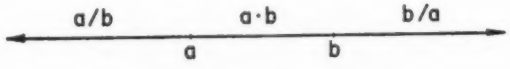


FIG. 6

a Euclidean line effected by its distinct points a, b (Fig. 6) is neatly expressed in terms of join and extension: line $ab = a/b \cup a \cdot b \cup b/a \cup a \cup b$.†

Our postulates merit a few remarks. (J1), $\cdots$, (J5) bear striking analogies with school algebra. Join—in contrast to multiplication in school algebra—may be described as a *many*-valued operation, since it associates with a pair of elements a set of elements rather than a single one. With this in mind (J1), $\cdots$, (J4) obviously correspond to familiar properties of multiplication and division in school algebra (or for that matter in an abelian group). (J5) is a generalization of the principle of cross-multiplication in a proportion (of school algebra or abelian group theory) to which it reduces when the four expressions involved reduce to single elements. This is a rather unexpected analogy between algebra

* It is interesting that Coxeter [2] uses this quotient notation to denote ray in a classical geometrical framework.

† Here we adopt the notational convention that in formulas involving $\cdot$, /, $\cup$, portions separated by $\cup$ signs are to be considered enclosed in parentheses.

and geometry. Although $aa=a$, the idempotent law for join, is well known in modern logic and Boolean algebra, (J6) is in a certain sense the most "geometric" of the postulates, since it is a combinatory property which holds for points and fails for numbers.

On the basis of (J1), $\cdots$, (J6) we shall develop a formal theory with many analogies to classical multiplicative algebra. The relations to classical algebra are interesting and stimulating. But a slavish imitation of classical algebra might distract us from our basic purpose: to study geometry by intrinsic methods. Our treatment should be judged primarily I think by its effectiveness in organizing, developing and generalizing classical geometrical material.

4. Formal properties of join. To prepare for the study of the basic geometric ideas, convex set, linear space and half-space, we develop the formalism of join and its inverse, extension, in this and the next section.

In the remainder of the paper, unless the contrary is indicated, we study an arbitrary system $(G, \cdot)$ which satisfies (J1), $\cdots$, (J6); we use $a, b, c, \cdots$ to denote elements of G and $A, B, C, \cdots$ subsets of G. Usually $(G, \cdot)$ is not explicitly mentioned in the theorems.*

We begin with a very simple "monotonic law."

THEOREM 1. *$A \subset B$ implies $AC \subset BC$ and $CA \subset CB$.*

Proof. We suppose $x \subset AC$ and deduce $x \subset BC$. By definition of join of sets (Sec. 3) $x \subset AC$ implies the existence of a, c satisfying

$$x \subset ac, \qquad a \subset A,\ c \subset C.$$

By hypothesis $a \subset A$ implies $a \subset B$. Thus $x \subset ac$, $a \subset B$, $c \subset C$. This implies by definition of join of sets $x \subset BC$. We infer $AC \subset BC$. Similarly we show $CA \subset CB$.

This says in effect that both sides of the "inequality" $A \subset B$ may be multiplied by C. We extend this to the multiplication of "inequalities" in

COROLLARY 1. *$A' \subset A$, $B' \subset B$ imply $A'B' \subset AB$.*

Proof. Multiplying $A' \subset A$ by B' and $B' \subset B$ by A (on the left) we obtain $A'B' \subset AB'$, $AB' \subset AB$, so that $A'B' \subset AB$.

In Corollary 1, let A' consist of a single element a and B' consist of b. Then we have

COROLLARY 2. *$a \subset A$, $b \subset B$ imply $ab \subset AB$.*†

As we have indicated earlier it is essential to operate with sets from the beginning. Thus we extend the associative law (J3) to sets.

* We assume and employ without specific reference the familiar intrinsic properties of equality and in addition, $a=b$ and $c=d$ imply $ac=bd$, and its analogue for sets.

† This is a direct consequence of the definition of AB; it is inserted here for convenience of formal presentation.

THEOREM 2. (ASSOCIATIVE LAW). $(AB)C=A(BC)$.

Proof. We show every element of each member is in the other. Suppose $x\subset(AB)C$. By the definition of join of sets (Sec. 3) this implies

$$x \subset yc, \tag{4.1}$$

where $y\subset AB$, $c\subset C$. Similarly we obtain

$$y \subset ab, \tag{4.2}$$

where $a\subset A$, $b\subset B$. By Theorem 1 we may multiply both members in (4.2) by "set" c, getting

$$yc \subset (ab)c = a(bc). \tag{4.3}$$

Then (4.1), (4.3) imply

$$x \subset a(bc). \tag{4.4}$$

By Corollary 2 above, $b\subset B$, $c\subset C$ imply $bc\subset BC$. This with $a\subset A$ implies by Corollary 1 above

$$a(bc) \subset A(BC). \tag{4.5}$$

Relations (4.4), (4.5) imply $x\subset A(BC)$. Similarly we show $x\subset A(BC)$ implies $x\subset(AB)C$ so that $(AB)C=A(BC)$.

As an immediate consequence of the theorem we have

$$a(b(cd)) = (ab)(cd)$$

taking a, b, cd as the three sets. This justifies formally our observation in Section 2 that the interior of a tetrahedron is the join of two opposite edges.

The commutative law for sets is easily proved and is left as an exercise.

THEOREM 3. $AB=BA$.

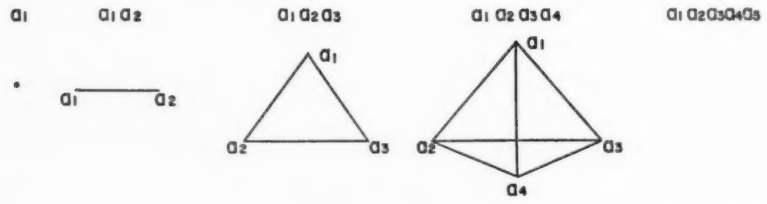
We do not want to restrict the application of join to two or three terms. So as in classical algebra we extend our operation to an arbitrary finite number of elements inductively:

$$a_1 \cdots a_{n+1} = (a_1 \cdots a_n)a_{n+1}.$$

Similarly we extend the operation to a finite number of sets. As in ordinary algebra the associative and commutative laws can be extended by induction to joins of n elements, but the proofs have to be modified somewhat since the join of two elements is not an element but a set. However it is easier to deal with sets rather than elements. For we have just seen that the basic associative and commutative laws do hold for sets, and certainly the join of two sets is a set. Thus we can apply the usual algebraic proofs (see, for example, [7], pp. 20–21) to obtain the associative and commutative laws for the join of n sets. Taking the

case where the sets reduce to single elements, we obtain the associative and commutative laws for the join of n elements.

This extension of join and its basic properties to n terms may seem dull and trivial. But consider for a moment the usual treatment of geometry in this regard. In classical plane and solid geometry the basic figures such as segments, triangles and tetrahedrons are studied on a visually motivated basis. When we study higher dimensional geometry it is hard to see how to generalize this treatment, and it is assumed conventionally that the only feasible procedure is to dispense with a direct (that is, conceptual) treatment of geometric objects and to assume the validity of coordinate or vectorial representation. The advantage is precisely that we can use n coordinates or combine n vectors essentially as easily as 2 or 3. So in our treatment the trivial-seeming extendability of the basic concept to n terms is of the essence: it fosters the direct study of geometric material of arbitrary dimension, while helping to liberate us from dependence on visual intuition. To make the point concrete, consider the following table which indicates the typical or nondegenerate Euclidean interpretation of some joins listed:



We have appended no diagram for the typical join of 5 elements! It is represented by the interior of the 4-dimensional analogue of a triangle or a tetrahedron, called a 4-dimensional simplex. The graphical representation in the second row breaks down at this stage—but the symbolic expression in the first row can be studied formally and abstractly, independently of the number of factors and so of dimension.*

We conclude this section with a remark on the application of the join operation to $\varnothing$, the void set. Although this is not of geometric importance, it is permitted since the definition of $A \cdot B$ involves no restriction on the sets A, B. The basic criterion of Section 3, $x \subset A \cdot B$ if and only if there exist a, b such that $x \subset ab$ and $a \subset A$, $b \subset B$, implies that $x \subset A \cdot \varnothing$ is always false, since $b \subset \varnothing$ is always false. Thus $A \cdot \varnothing = \varnothing$. Similarly $\varnothing \cdot B = \varnothing$.

5. The inverse operation. We are now ready to study the properties of extension, the operation inverse to join. Following a familiar pattern we extend this operation to sets.

* This suggests a questionable piece of humor to make a serious point: 4-dimensional space is only 25% harder to study than 3-dimensional space.

DEFINITION. $A/B = \bigcup_{a \subset A,\, b \subset B} (a/b)$. Thus $x \subset A/B$ if and only if $x \subset a/b$, where $a \subset A$, $b \subset B$.

It follows from this definition that $A/\varnothing = \varnothing = \varnothing/A$ by the argument of the last paragraph of Section 4.

In Euclidean geometry, A/B will be the set of points each of which is in an extension, a/b, of a point of A from a point of B. For example if A is a single point not in the plane of circle B, A/B is a one-sheeted conical surface with vertex A, which is open in the sense of not containing A (Fig. 7).

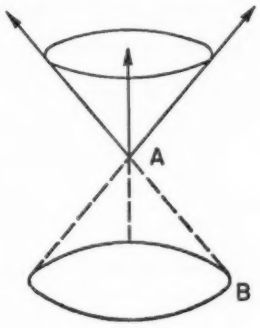


FIG. 7

The proof of Theorem 1, the monotonic law for join, is valid for the inverse operation as well and we may assert

THEOREM 4. *$A \subset B$ implies $A/C \subset B/C$ and $C/A \subset C/B$.*

COROLLARY 1. *$A' \subset A$, $B' \subset B$ imply $A'/B' \subset A/B$.*

COROLLARY 2. *$a \subset A$, $b \subset B$ imply $a/b \subset A/B$.*

Our most important technical procedure involves a simple calculus of intersection relations. Given $A \approx B$, an intersection relation between two sets, we can immediately infer other intersection relations, for example if $X \neq \varnothing$, $AX \approx BX$ or $A/X \approx B/X$ or even $X/A \approx X/B$. These correspond to simple geometric properties of intersecting sets. Moreover the relation between join and its inverse, extension, can be expressed by a very simple and convenient intersection algorithm.

THEOREM 5. *$A \approx BC$ if and only if $A/B \approx C$.*

Proof. Suppose $A \approx BC$. This means that there exists a such that $a \subset A$, $a \subset BC$. The latter implies $a \subset bc$ where $b \subset B$, $c \subset C$. Thus $bc \supset a$ and by definition of a/b we have $c \subset a/b$. This with $a \subset A$, $b \subset B$ implies, by definition of A/B, that $c \subset A/B$. Since we already know $c \subset C$ we conclude $A/B \approx C$.

For the converse we reason similarly. Suppose $A/B \approx C$. Then $c \subset A/B$,

$c \subset C$ for some c. Thus $c \subset a/b$ where $a \subset A$, $b \subset B$. By definition of a/b we have $bc \supset a$. Thus $BC \supset a$ and since $A \supset a$ we conclude $A \approx BC$.

Permitting A, B, C to reduce to single elements, we have the

COROLLARY. *$a \approx bc$ if and only if $a/b \approx c$.*

Note here the algorithmic advantage of using $\approx$, which is suggestive of equality—without it we would have a sort of inequality principle: $a \subset bc$ if and only if $a/b \supset c$.

The transposition law (J5) is easily extended to sets.

THEOREM 6. *$A/B \approx C/D$ implies $AD \approx BC$.*

Proof. By hypothesis there exists x such that

$$x \subset A/B, \qquad x \subset C/D.$$

Hence by definition

$$x \subset a/b, \qquad x \subset c/d,$$

where a, b, c, $d \subset A$, B, C, D, respectively. Thus $a/b \approx c/d$ and (J5) yields $ad \approx bc$. By Theorem 1, Corollary 2,

$$AD \supset ad, \qquad BC \supset bc$$

and we infer $AD \approx BC$.

We continue with somewhat deeper properties of extension or division. Glancing back at (J1), $\cdots$, (J6) we see that we have not postulated the existence of an identity element or of inverses of elements. Consequently we cannot hope to *reduce* division to multiplication in the ordinary way, $a/b = a(1/b)$. To compensate for this we *relate* division to multiplication by "mixed associative principles" like the following:

THEOREM 7. $a/bc = (a/b)/c$.*

Proof. We show every element of each member is in the other. Suppose $x \subset a/bc$, that is, $x \approx a/bc$. Then

$$\begin{aligned} x(bc) &\approx a && \text{(Th. 5)},\\ (xc)b &\approx a && \text{(J2), (J3)},\\ xc &\approx a/b && \text{(Th. 5)},\\ x &\approx (a/b)/c && \text{(Th. 5)}. \end{aligned}$$

Conversely we suppose $x \approx (a/b)/c$ and retrace the steps to complete the proof.

* Here a/bc stands for $a/(bc)$. In general in formulas involving $\cdot$, /, we eliminate excess parentheses by the agreement that portions separated by / signs are to be considered enclosed in parentheses. For example $(A/B)C/DE$ denotes $((A/B)C)/(DE)$.

Using the method employed in Theorem 2 to extend the associative law from elements to sets, we easily get the

COROLLARY. $A/BC=(A/B)/C$.

You might expect that the other laws of school algebra which link multiplication and division

$$a/(b/c) = ac/b, \qquad a(b/c) = ab/c$$

hold here also. This is not so. It would be too boring if there were no divergencies between ordinary algebra and our geometric algebra! However, generalizations of these laws, replacing equality by containment, do hold and can be derived by the method used above:

THEOREM 8. (a) $a/(b/c)\subset ac/b$; (b) $a(b/c)\subset ab/c$.

These can be extended as usual to sets:

COROLLARY. (a) $A/(B/C)\subset AC/B$; (b) $A(B/C)\subset AB/C$.

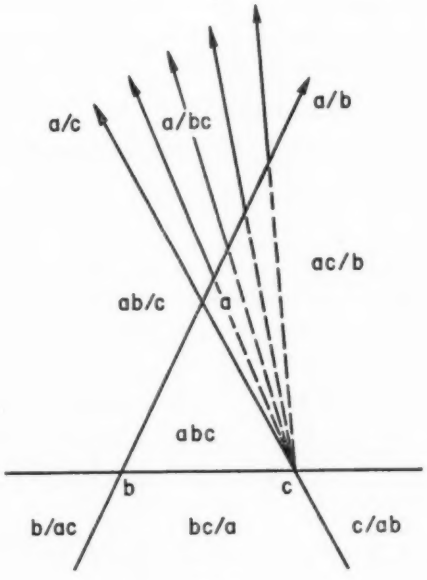


FIG. 8

You must be curious about the concrete geometric significance of Theorems 7, 8. Let a, b, c be the vertices of a triangle in a Euclidean plane (Fig. 8). Observe that the seven regions into which the sidelines of $\triangle abc$ separate the plane are

represented as certain fractional expressions involving a, b, c. For example, a/bc is the interior of the angle formed by rays a/b, a/c. Now note that this region is covered by the rays which emanate from the points of a/b and are directed away from c. Thus a/bc and $(a/b)/c$ are the same region and Theorem 7 is verified. Theorem 8 is easily verified in the same example.

6. Convex sets. Finally we are prepared to study convex sets. We formulate the concept in join language as follows:

DEFINITION. *A is* ***convex*** *or* ***multiplicatively closed*** *if $A \supset x, y$ implies $A \supset xy$.*

Observe that G, the basic set, is convex and that each element a forms a convex set since by (J6), $a = aa \supset aa$. Note that $\varnothing$ also is convex since the definition imposes no restriction on it.

The definition is easily reformulated in terms of join of sets to yield

THEOREM 9. *A is convex if and only if* (a) $A \supset AA$; *or* (b) $A = AA$.

Proof. (a) Suppose A convex. Then $A \supset xy$ for $x \subset A$, $y \subset A$. Thus $A \supset AA$, which is the union of all such sets xy. Conversely if $A \supset AA$ certainly $A \supset xy$, for $x \subset A$, $y \subset A$, and is convex.

(b) Observe for any set A that $A \subset AA$, for if $x \subset A$ then by (J6), $x = xx \subset AA$. Thus conditions (a), (b) are equivalent and the theorem holds.

COROLLARY 1. *$a_1a_2 \cdots a_n$ is convex.*

Proof. By the generalized associative and commutative laws and (J6)

$$(a_1a_2 \cdots a_n)(a_1a_2 \cdots a_n) = (a_1a_1)(a_2a_2) \cdots (a_na_n) = a_1a_2 \cdots a_n$$

and the result follows by (b) of the theorem.

In Euclidean geometry this justifies the convexity of the interior of a triangle or a tetrahedron or in general of any n-simplex.

By definition, convexity is closure under join of elements—it implies closure under join of sets:

COROLLARY 2. *If A is convex and $A \supset X$, Y then $A \supset XY$.*

Proof. $A \supset X$, $A \supset Y$ can be combined to yield $AA \supset XY$ or $A \supset XY$.

We naturally are concerned about the application of familiar operations to convex sets.

THEOREM 10. *If A, B are convex then $A \cap B$, AB, and A/B are convex.*

Proof. The result for $A \cap B$ is immediate since $A \supset x, y$ and $B \supset x, y$ imply $A, B \supset xy$ so that $A \cap B \supset xy$.

For AB we have using Theorem 9(b) $(AB)(AB) = (AA)(BB) = AB$ so that AB is convex by Theorem 9(b).

For A/B it is a bit more complex and interesting. We have

$$
\begin{aligned}
(A/B)(A/B) &\subset (A/B)A/B && \text{(Th. 8, Corol. (b))}\\
&= A(A/B)/B && \text{(Th. 3)}\\
&\subset (AA/B)/B && \text{(Th. 8, Corol. (b); Th. 4)}\\
&= AA/BB && \text{(Th. 7, Corol.)}\\
&= A/B && \text{(Th. 9(b))}
\end{aligned}
$$

and A/B is convex by Theorem 9(a).

An illustration of the last principle: In Euclidean geometry let A be a point and B the interior of an ellipse whose plane does not contain A, then A/B is the interior of an elliptic cone with vertex A and is convex.

7. Convexifying sets. Suppose we have a set which is not convex. It is natural then to try to *convexify* it—that is to make it convex in the simplest possible way. For example, in Euclidean geometry consider the set (a, b) composed of the distinct points a, b. If we adjoin to (a, b) each point of ab we obtain the closed interval with endpoints a, b which is convex. Clearly this is the simplest way to convert (a, b) into a convex set, since any convex set containing a

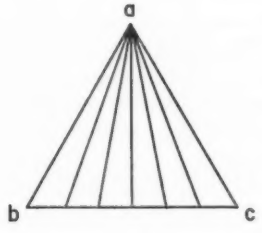


FIG. 9

and b must by definition contain ab. Similarly, if a, b, c are the vertices of a Euclidean triangle the simplest way to make (a, b, c) convex is to adjoin to it each point of ab, bc, ac and abc, thus obtaining the closed triangular region with vertices a, b, c (Fig. 9).

These illustrations are typical of the general theory as we show now.

THEOREM 11. *Consider the finite set* $(a_1, \cdots, a_n)$. *Let* S *be the union of joins of* $a_1, \cdots, a_n$ *taken one or more at a time*:

$$S = a_1 \cup a_2 \cup \cdots \cup a_n \cup a_1a_2 \cup a_1a_3 \cup \cdots \cup a_{n-1}a_n \cup \cdots \cup a_1a_2 \cdots a_n.$$

Then S *is the only set which satisfies the following properties*:

(a) S *is convex*;
(b) $S \supset (a_1, \cdots, a_n)$;
(c) *If* X *is convex and* $X \supset (a_1, \cdots, a_n)$ *then* $X \supset S$.

Proof. (a) Suppose $x, y \subset S$. Then x, y belong to joins of the a's. For example, suppose $x \subset a_1a_3a_5$, $y \subset a_2a_6a_7$. Combining these as inequalities (Th. 1, Corol. 2),

$$xy \subset (a_1a_2a_5)(a_2a_6a_7) = a_1a_2a_5a_6a_7.$$

Thus $xy \subset S$ and S is convex by definition.

(b) This is trivial by definition of S.

(c) Suppose X convex and $X \supset (a_1, \cdots, a_n)$. By Theorem 9, Corollary 2, used repeatedly, X contains any join of the a's and so $X \supset S$.

To prove uniqueness, suppose S' satisfies (a), (b), (c). Letting $X = S'$ in (c) we have $S' \supset S$. Similarly $S \supset S'$ so that $S' = S$.

Thus in constructing S we have converted the finite set $(a_1, \cdots, a_n)$ into a convex set in the simplest possible way, since by (c) any other convex set containing $(a_1, \cdots, a_n)$ must be "larger than" S. This suggests a precise formulation of the concept of simplest convex set containing a given (finite or infinite) set:

DEFINITION. *Let set A be given. Let S be the only set which satisfies the following properties*:

(a) *S is convex*;

(b) $S \supset A$;

(c) *If X is convex and $X \supset A$ then $X \supset S$.*

Then we call S the ***least convex set*** *containing A, the* ***convex set generated*** *or* ***spanned by*** *A or the* ***convex closure*** *of A and denote it by $[A]$. We write $[(a_1, \cdots, a_n)]$ simply $[a_1, \cdots, a_n]$.*

Observe that $[A] = A$ if A is convex; in particular $[a] = a$. Restating Theorem 11 we have the

COROLLARY. $[a_1, \cdots, a_n] = a_1 \cup \cdots \cup a_n \cup a_1a_2 \cup \cdots \cup a_{n-1}a_n \cup \cdots \cup a_1a_2 \cdots a_n$.

For any set A, not necessarily finite, it is not hard to show a similar result, that $[A]$ exists and is the union of all joins $a_1 \cdots a_r$, where the a's are in A. Thus not merely are finite joins convex (Th. 9, Corol. 1) but they are the building blocks of which all convex sets are composed.

In Euclidean geometry the notion convex closure covers the ideas of closed interval, closed convex polygon and closed convex polyhedron. For example, if a, b are distinct points, $[a, b]$ is the closed interval with endpoints a, b. Further if a, b, c, d are points of a Euclidean plane no one of which is in the join of the other three, $[a, b, c, d]$ is the closed convex polygonal region whose vertices are a, b, c, d (Fig. 10(a)). If however one of the points is in the join of the others, say $d \subset abc$, then $[a, b, c, d]$ reduces to $[a, b, c]$ which in general is a closed triangular region (Fig. 10(b)). Observe that if a, b, c, d are in *general* position, that is do not coplane, $[a, b, c, d]$ is a 3-dimensional simplex and the terms in

its expansion given in the last corollary are disjoint. This is not true if a, b, c, d coplane.

Finally let me remark that the concept of triangle, which is visually inspired and historically grounded in practical problems of measurement, might in the present context have been discovered by a blind student. It forces itself on our attention as one of the simplest kinds of convex set. In effect a visual-pragmatic motivation of the idea is replaced by a conceptual one.

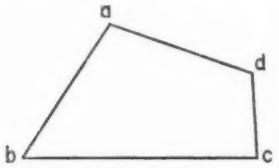


FIG. 10(a)

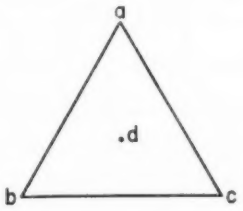


FIG. 10(b)

8. Linear sets. Linear set—or linear space as it is usually called—is a generic idea designed to cover the basic "linear" (or uncurved) figures: endless line, plane, 3-space, etc. It is defined in Euclidean geometry (or in other linear geometries) as a set of points containing the (endless) line joining each two of its points.

This puts us face to face with a disturbing question: How in an abstract join system can we introduce the notion line? We have "join" to correspond to segment, "quotient" to correspond to ray, but it is hard to see how to give a simple and natural construction for line. The conventional definition in the foundations of Euclidean geometry ([2], p. 161) expressed in join language is

$$\text{line } ab = ab \cup a/b \cup b/a \cup a \cup b \qquad (a \neq b). \tag{8.1}$$

But this is precisely what we are trying to avoid: namely to assume, in the guise of a definition, an apparently restrictive form—in effect another postulate —without intrinsic motivation, merely on the basis of visual intuition or practical necessity.

So we lay aside for the moment the question of line, and try to define *linear set* in terms of the ideas at our disposal. This is easier! The clue lies in the Euclidean situation. Clearly here a point set containing a and $b (a \neq b)$ will contain line ab if and only if it contains segment ab and rays a/b, b/a. Thus a Euclidean point set is linear if and only if it contains the join and the extension of any two of its points, or equivalently, is closed under the operations join and extension. Thus we adopt formally the

DEFINITION. *A is* ***linear*** *(or a* ***linear space*** *or a* ***linear subspace*** *of G) if $A \supset x, y$ implies $A \supset xy, x/y$.*

A linear set is a convex set which is so to speak "fully extended." Observe that each individual element is linear, since $a = aa = a/a$ by (J6). Also G and $\varnothing$ are linear.

We begin with an analogue of Theorem 9, which is proved by the method of Theorem 9.

THEOREM 12. *A is linear if and only if it is convex and* (a) $A \supset A/A$, *or* (b) $A = A/A$.

Corresponding to Corollary 2 of Theorem 9 we have the

COROLLARY. *If A is linear and $A \supset X$, Y then $A \supset XY$, X/Y.*

Corresponding to Theorem 10 we have merely

THEOREM 13. *If A, B are linear then $A \cap B$ is linear.*

If A and B are linear, AB and A/B in general are not linear—for example, in Euclidean geometry let A, B be parallel lines, or A a line and B a point not in A. Evidently it is harder to construct linear sets than convex ones—however the following theorem gives an important method.

THEOREM 14. *If A is convex A/A is linear.*

Proof. A/A is convex by Theorem 10. Further we have

$$\begin{aligned}
(A/A)/(A/A) &\subset (A/A)A/A && \text{(Th. 8, Corol. (a))} \\
&= A(A/A)/A && \text{(Th. 3)} \\
&\subset (AA/A)/A && \text{(Th. 8, Corol. (b); Th. 4)} \\
&= AA/AA && \text{(Th. 7, Corol.)} \\
&= A/A. && \text{(Th. 9)}
\end{aligned}$$

Thus A/A is linear by Theorem 12(a).

Note the similarity between the argument above and that of the last part of Theorem 10. To illustrate the result let A be the interior of a Euclidean sphere. Then A/A contains all rays which are prolongations of segments in A and clearly is the 3-space containing A.

In general A/A might be called the "complete extension" of A, since it is the aggregate of all extensions a_1/a_2 for a_1, $a_2 \subset A$. Then Theorem 14 asserts that the complete extension of a convex set is linear.

The most important special case of the theorem occurs when $A = a_1 \cdots a_n$, which we know is convex by Theorem 9, Corollary 1. Thus we have the

COROLLARY. *$a_1 \cdots a_n/a_1 \cdots a_n$ is linear.*

This enables us to construct a linear set from any finite set. For example in higher-dimensional Euclidean geometry if a_1, a_2, a_3, a_4, a_5 are not in a 3-space then $a_1a_2a_3a_4a_5/a_1a_2a_3a_4a_5$ is a 4-space.

We saw in the paragraph following Theorem 13, that if A, B are linear A/B need not be so. However A/B is linear in the face of the additional condition $A \approx B$.

THEOREM 15. *If A, B are linear and $A \approx B$ then A/B is linear.*

We shall dispense with the proof which is somewhat similar to, but more difficult than, those of Theorems 10 and 14.* The result is very important since it supplements Theorem 14 as a method of constructing linear sets. Despite its

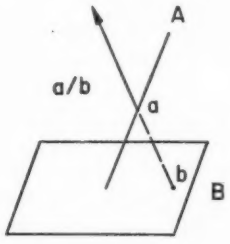


FIG. 11

generality (it involves no dimensional restriction) it has strong intuitive geometric content. For example in a Euclidean 3-space G, let line A meet plane B in a single point. Observe how the extensions a/b for $a \subset A$, $b \subset B$ cover G (Fig. 11).

We conclude this section with an important theoretical point involving linear sets. Suppose S is a linear set of join system $(G, \cdot)$. Since S is closed under the operation $\cdot$ it is natural to consider the system $(S, \cdot)$. Is it a join system? Clearly $(S, \cdot)$ satisfies (J1). The formal laws (J2), (J3) and the first part of (J6) hold in $(G, \cdot)$ and so certainly in $(S, \cdot)$. To consider the remaining postulates we must define extension in $(S, \cdot)$. But S is closed under /, the extension operation in $(G, \cdot)$. Hence the extension operation in $(S, \cdot)$ must be / applied merely to elements of S. It is now not hard to see that (J4), (J5), (J6) hold in $(S, \cdot)$. Thus $(S, \cdot)$ is indeed a join system, which we signalize by saying $(S, \cdot)$ is a *subsystem* of $(G, \cdot)$ or simply S is a *subsystem* of G. To summarize: *Any linear set S may be considered to be a join system in its own right to which our theory can be applied.* This of course is analogous to the subsystem concept in modern algebra, as exemplified by subgroups of a group, subfields of a field, etc.

9. Linearizing sets. Suppose we have a set which is not linear. It is natural then to try to *linearize* it—that is to make it linear in the simplest possible way. In Euclidean geometry, when we form the line determined by points a, b or

* For proof see [10], p. 352, Theorem 5, which covers Theorem 15. Note that [10] adopts an additive notation for join rather than the multiplicative one used here, and a different definition for joins involving the void set.

the plane determined by a, b, c, we are in effect linearizing set (a, b) or (a, b, c). In these cases (and in Euclidean geometry in general) an analogy with the process of convexification of Section 7 holds. Thus to linearize (a, b) we merely adjoin to it ab and a/b and b/a (see Fig. 6). To linearize (a, b, c) we adjoin the joins ab, bc, ac, abc, the extensions $a/b, b/a, b/c, c/b, a/c, c/a$ and the six "mixed" expressions $a/bc, b/ac, c/ab, ab/c, bc/a, ac/b$ (see Fig. 8).

This process of "linearizing by adjunction" is not valid in our abstract join theory. The corollary to Theorem 14 provides a clue to the correct result:

THEOREM 16. *Consider the finite set* $(a_1, \cdots, a_n)$. *Let* $S = a_1 \cdots a_n/a_1 \cdots a_n$. *Then* S *is the only set which satisfies the following properties*:

(a) S *is linear*;

(b) $S \supset (a_1, \cdots, a_n)$;

(c) *If* X *is linear and* $X \supset (a_1, \cdots, a_n)$ *then* $X \supset S$.

Proof. (a) Theorem 14 Corollary.

(b) We have to show $a_i \approx a_1 \cdots a_n/a_1 \cdots a_n$, $(1 \leqq i \leqq n)$. This is valid since it is equivalent by Theorem 5 to $a_i(a_1 \cdots a_n) \approx a_1 \cdots a_n$, which is equivalent by J6 to the truism $a_1 \cdots a_n \approx a_1 \cdots a_n$.

(c) Let X be linear, $X \supset (a_1, \cdots, a_n)$. By repeated application of Theorem 9, Corollary 2, $X \supset a_1 \cdots a_n$ and by Theorem 12, Corollary, $X \supset a_1 \cdots a_n / a_1 \cdots a_n$.

Uniqueness follows as in Theorem 11 and the proof is complete.

Thus we have linearized any *finite* set. Naturally we adopt the

DEFINITION. *Let set* A *be given. Let* S *be the only set which satisfies the following properties*:

(a) S *is linear*;

(b) $S \supset A$;

(c) *If* X *is linear and* $X \supset A$ *then* $X \supset S$.

Then we call S *the* **least linear set** *containing* A, *the linear set* **generated** *or* **determined by** A *or the* **linear closure** *of* A, *and denote it by* $\{A\}$. *We write* $\{(a_1, \cdots, a_n)\}$ *as* $\{a_1, \cdots, a_n\}$.

Note $\{A\} = A$ if A is linear, in particular $\{a\} = a$, $\{\varnothing\} = \varnothing$.

COROLLARY. $\{a_1, \cdots, a_n\} = a_1 \cdots a_n/a_1 \cdots a_n$.

Now we can easily answer the question: What should a line be? Clearly line ab should contain the distinct elements a and b, should be linear, and should be the least or simplest set with these properties. Thus we adopt the

DEFINITION. *If* $a \neq b$, *line* ab *is* $\{a, b\}$.

By the corollary above we have the formula

$$\text{line } ab = ab/ab.$$

Observe that by definition $\{a, b\}$ contains a, b and so ab, a/b, b/a. This gives

$$\{a, b\} \supset ab \cup a/b \cup b/a \cup a \cup b, \tag{9.1}$$

a simple but important relation which connects, when $a \neq b$, the conventional definition of line (8.1), and the one we have adopted.*

Does $\{A\}$ exist for an arbitrary set A, not necessarily finite? To answer this suppose X linear and $X \supset A$. Then X is convex, and $X \supset [A]$ by definition of $[A]$. Further $X \supset [A]/[A]$ by Theorem 12, Corollary. $[A]/[A]$ is linear by Theorem 14. By (J6) and the definition of $[A]$, $[A]/[A] \supset [A] \supset A$. Thus $[A]/[A]$ satisfies (a), (b), (c) of the definition of $\{A\}$. Uniqueness follows easily (as in the proof of Theorem 11) and we infer

THEOREM 17. $\{A\} = [A]/[A]$.

That is, to linearize A we merely convexify it and take the "complete extension" of the result.

The notion of linear closure can be extended to several sets. The importance of this is indicated by the use in school geometry of phrases such as "the plane determined by a line and a point," or "the plane determined by two lines." In higher-dimensional Euclidean geometry we should like to refer, for example, to the linear space determined by two planes which intersect in a single point—it would of course be a 4-space. Thus for two sets A, B we define $\{A, B\}$, the linear set *generated* or *determined by* A and B or the *linear closure* of A and B, to be the unique set S which satisfies

(a) S is linear;

(b) $S \supset A, B$;

(c) *If* X *is linear and* $X \supset A, B$ *then* $X \supset S$.

The particular case where A, B are linear and $A \approx B$ is quite important. We recall (Th. 15) that in this case A/B is linear. Further we can show $A/B \supset A, B$.† Clearly if linear set $X \supset A, B$ then $X \supset A/B$. Uniqueness follows as usual and we infer

THEOREM 18. *If* A, B *are linear and* $A \approx B$ *then* $\{A, B\} = A/B$.‡

For arbitrary A, B we can obtain a formula for $\{A, B\}$ (see, for example, [10], p. 351, Th. 4) by suitably generalizing the argument of Theorem 17. However the *existence* of $\{A, B\}$ is easily shown by a kind of argument which is familiar in modern algebra. Let S be the intersection of all linear sets X such that $X \supset A, B$. Then S is linear by the argument of Theorem 13. $S \supset A, B$; and if linear $X \supset A, B$, clearly $X \supset S$. We infer $S = \{A, B\}$. The same kind of argu-

* The conventional definition of line is not equivalent to ours, see Appendix, Section E for an example of a join system in which (9.1) is not an equality.

† For $A/B \supset B$ see proof of [10], p. 352, Theorem 5. For $A/B \supset A$ it suffices to show $A/a \supset A$ if $a \subset A$, which follows from the convexity of A.

‡ See [10], p. 352, Theorem 5.

ment could have been used to prove the existence of $[A]$ and $\{A\}$ (see, for example, [7], p. 30, Sec. 11).

10. Half-spaces and congruence relations. We consider now the theory of half-spaces: rays or half-lines, half-planes, half-3-spaces, etc. These form a subclass of the convex sets which is next in importance to the linear sets. As the name suggests, half-spaces are often defined in terms of separation of linear spaces. Thus in Euclidean geometry a half-plane is either of the two sets into which a line separates a plane, (see [4], p. 68, Def. 2.1; or [2], p. 163). This characterization is not appropriate here, since our postulates are too weak to imply such separation—even in Euclidean geometry it would force us to prove a rather complicated separation theorem ([4], p. 68) before discussing the relatively simple notion of half-space. However, an analysis of the Euclidean situation yields a simple "congruence" or equivalence relation, which has immediate geometric content and leads directly to the notion of half-space.

First we consider a specific Euclidean example. Let M be a line in a Euclidean 3-space and a a point not in M (Fig. 12). We know that there is a unique half-plane with edge M containing a, but instead of trying to construct this we ask:

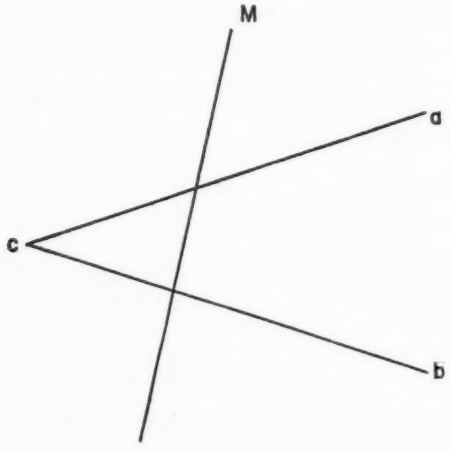


FIG. 12

When are points a and b in the same half-plane with edge M, or simply when are a and b on the same side of M? More precisely stated, we seek a criterion, in join terminology, for the relation of a and b to M if a and b are on the same side of M.

Suppose then a, b are on the same side of M. Instead of trying to relate a and b directly, we use the principle that if a and b are on the same side of M they are both on the opposite side of M from some point c. Thus there exists a

point c (see Fig. 12) such that the joins of c to a and b meet M; that is, we have

$$ac \approx M, \qquad bc \approx M. \tag{10.1}$$

Solving the relations in (10.1) for c we get

$$c \approx M/a, \qquad c \approx M/b, \tag{10.2}$$

so that $M/a \approx M/b$. Applying the transposition law for sets (Th. 6) we get

$$Ma \approx Mb. \tag{10.3}$$

Conversely suppose (10.3) and $a, b \not\subset M$. Then for some point d, $d \approx Ma$, $d \approx Mb$ (see Fig. 13), so that $d \approx m_1 a$, $d \approx m_2 b$, where $m_1, m_2 \subset M$. We infer that a and b are on the same side of M, since each is on the same side of M as d. To summarize: Suppose $a, b \not\subset M$. Then a, b are on the same side of M if and only if they satisfy (10.3).

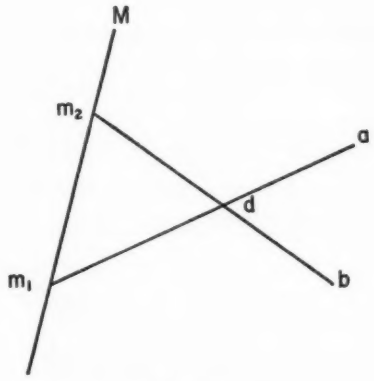


FIG. 13

This suggests the importance, in our abstract join theory, of studying (10.3) as a relation between a and b when M is a given linear set. Since the relation depends on the choice of M, and a, b are equivalent in a certain sense relative to M, I adopt the algebraic terminology of "congruence with respect to a modulus" to describe it.

DEFINITION. *Let M be a nonvoid linear set. Then $a \equiv b \pmod{M}$, read a **is congruent to** b **modulo** M, means $aM \approx bM$.*

Although the relation $a \equiv b \pmod{M}$ is suggested by the Euclidean relation, a and b are on the same side of linear space M, it is not exactly equivalent to this, since it also covers the "degenerate" case where a and b are in M. Precisely, $a \equiv b \pmod{M}$ holds in a Euclidean geometry if and only if (a) a and b are on the same side of M or (b) a and b are in M. We show this for the case M is a

line. We saw above that (a) is equivalent to $a \equiv b(\text{mod } M)$ provided $a,b \not\subset M$. Suppose then one of a, b, say a, is in M. Let $a \equiv b(\text{mod } M)$. Then $aM \approx bM$ which implies $b \approx aM/M \subset M$, and both a and b are in M. Conversely suppose $a, b \subset M$. Then $aM, bM \supset ab$ so that $aM \approx bM$ and $a \equiv b(\text{mod } M)$. Thus our statement is justified in the given case.

We proceed to study the properties of congruence mod M in an abstract join system. It is not surprising that congruence mod M is an equivalence relation:

THEOREM 19. (a) $a \equiv a(\text{mod } M)$; (b) $a \equiv b(\text{mod } M)$ implies $b \equiv a(\text{mod } M)$; (c) $a \equiv b(\text{mod } M)$, $b \equiv c(\text{mod } M)$ imply $a \equiv c(\text{mod } M)$.

Proof. (a), (b) are immediate from the definition.

To prove (c): We are given $aM \approx bM$, $bM \approx cM$. Solving for b, $b \approx aM/M$, $b \approx cM/M$; eliminating b, $aM/M \approx cM/M$. Applying the transposition law for sets (Th. 6), $aMM \approx cMM$. Since M is linear it is convex. Thus $MM = M$ by Theorem 9(b) so that $aM \approx cM$ and $a \equiv c(\text{mod } M)$.

We know in the arithmetic of integers that the congruences $a \equiv b(\text{mod } m)$, $c \equiv d(\text{mod } m)$ can be added to yield $a+c \equiv b+d(\text{mod } m)$. Can geometric congruences be combined similarly under the operation join or multiplication? This would require us to find an appropriate extension of the notion congruence from elements to *sets*. The answer is implicit in the following

LEMMA. *Suppose $a \equiv b(\text{mod } M)$ and c is any element. Then each element of ca is congruent modulo M to some element of cb.*

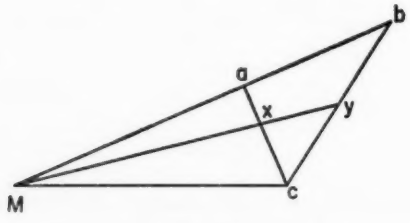


FIG. 14

Proof. Let $x \subset ca$. By hypothesis $aM \approx bM$. (See Fig. 14 which illustrates the Euclidean case where M is a point.) Solving the last relation for a we get $a \subset bM/M$. Multiplying both sides by c and applying Theorem 8, Corollary (b),

$$ca \subset c(bM/M) \subset cbM/M.$$

Thus $x \approx cbM/M$ so that $xM \approx cbM$, $xM/M \approx cb$. Consequently $xM/M \approx y$ and $y \approx cb$ for some y. Transposing M in the former relation yields $xM \approx yM$ so that $x \equiv y(\text{mod } M)$ and the Lemma holds.

To answer our question on multiplication of congruences, we consider first

the special case: Does $a \equiv b \pmod{M}$ imply, in a suitable sense, $ca \equiv cb \pmod{M}$? Suppose $a \equiv b \pmod{M}$. Then by the lemma each element of ca is congruent mod M to some element of cb. By symmetry each element of cb is congruent mod M to some element of ca. This rather close relation of the sets ca and cb we symbolize by: $ca \equiv cb \pmod{M}$. In general we adopt the

DEFINITION. *Let M be a nonvoid linear set. Then $A \equiv B \pmod{M}$ means that for each $x \subset A$ there is a $y \subset B$ such that $x \equiv y \pmod{M}$ and vice versa.*

Observe that Theorem 19 is valid for congruence of sets. That is, the relation congruence mod M is an equivalence when applied to sets just as well as to elements—this tends to confirm the appropriateness of our definition.

Our discussion is summarized in

THEOREM 20. *$a \equiv b \pmod{M}$ implies $ca \equiv cb \pmod{M}$.*

This easily yields the basic result on combining congruences:

COROLLARY. *$a \equiv b \pmod{M}$, $c \equiv d \pmod{M}$ imply $ac \equiv bd \pmod{M}$.*

Proof. Multiplying the given congruences by c, b respectively we have

$$ac \equiv bc \pmod{M}, \qquad bc \equiv bd \pmod{M},$$

and the result follows since congruence mod M is an equivalence for sets.

Every arithmetic congruence relation has an *additive identity element*, since $a+m \equiv a \pmod{m}$. We have an analogue:

THEOREM 21. *$am \equiv a \pmod{M}$ provided linear set $M \supset m$.*

Proof. Suppose $x \subset am$. Then

$$xM \subset amM = a(mM) \subset aM.$$

Certainly $xM \approx aM$, so that $x \equiv a \pmod{M}$. The last definition yields $am \equiv a \pmod{M}$.

COROLLARY. *Let M be linear and $m \subset M$. Then $x \equiv m \pmod{M}$ if and only if $x \subset M$.*

Proof. $x \equiv m \pmod{M}$ implies $xM \approx mM$ and $x \approx mM/M \subset M$. Conversely suppose $x \subset M$. The theorem implies

$$xm \equiv x \pmod{M}, \qquad xm \equiv m \pmod{M}$$

so that $x \equiv m \pmod{M}$.

Since congruence modulo M is an equivalence relation (Th. 19), we naturally introduce its associated *equivalence classes* ([1], pp. 155, 156).

DEFINITION. *The set of x which satisfies $x \equiv a \pmod{M}$ is called the* **congruence set** mod M **determined by** a *and is denoted by $(a)_M$. Briefly we call $(a)_M$ the* **coset of M determined by** a.

We employ the term coset not merely as an abbreviation of congruence set, but to signalize the striking analogies of this concept to the notion coset in classical group theory ([1] p. 142). For geometrical suggestiveness and for application to Euclidean (or any linearly ordered) geometry we call $(a)_M$ the *half-space with edge M determined by a* provided $a \not\subset M$.*

We continue with properties of cosets. Directly from Theorem 19 we have ([1], p. 156, Lemma 2):

THEOREM 22. *The cosets of linear set M form a partition of G.*

It is important to have a formula for a coset:

THEOREM 23. $(a)_M = aM/M$.

Proof. $x \subset (a)_M$ is equivalent to $x \equiv a \pmod{M}$. By definition this is equivalent to $xM \approx aM$ and so to $x \approx aM/M$.

COROLLARY. *$(a)_M$ is convex.*

Proof. By Theorem 10, aM/M is convex since a, M are.

Restating Theorem 21, Corollary, in coset language we have:

THEOREM 24. *Let M be linear and $m \subset M$. Then $(m)_M = M$.*

Thus the "modulus" M is a coset.

Just as a Euclidean half-space (say a half-line or half-plane) has an opposite half-space, so every coset has an "opposite" coset:

DEFINITION. *Cosets A, B of M are **opposite** (or A is **opposite** to B) if there exist $a \subset A$, $b \subset B$ such that $ab \approx M$.*

The defining property extends to each pair of elements chosen from the respective cosets:

THEOREM 25. *Let A, B be opposite cosets of M. Then $x \subset A$, $y \subset B$ imply $xy \approx M$.*

Proof. By definition $A \supset a$, $B \supset b$ and $ab \approx M$. Then A and $(a)_M$ have a in common and $A = (a)_M$ by Theorem 22. Similarly $B = (b)_M$. Consequently $x \subset (a)_M$, $y \subset (b)_M$ so that

$$x \equiv a \pmod{M}, \qquad y \equiv b \pmod{M}.$$

Combining these congruences (Th. 20, Corol.), $xy \equiv ab \pmod{M}$. Let $m \subset ab$, M. By definition of congruent sets, $m \subset ab$ implies the existence of $m' \subset xy$ such that $m' \equiv m \pmod{M}$. By Theorem 21, Corollary, $m' \subset M$. Thus $xy \approx M$ and the proof is complete.

COROLLARY 1. *Let A, B be opposite cosets of M. Then $A \cap B = \varnothing$ or $A = B = M$.*

* This is to some extent a misnomer, since it suggests a separation theorem which does not hold in all join systems, see Appendix Section G.

Proof. Suppose $A \cap B \neq \varnothing$. Then $A = B$ (Th. 22). Let $p \subset A$. Then $p \subset B$ and the theorem implies $p = pp \approx M$. Thus $A = B = M$ by Theorem 22.

COROLLARY 2. *M is the only coset of M which is its own opposite.*

Proof. M is opposite to M by definition, since if $m \subset M$ then $mm = m \subset M$. Uniqueness follows from Corollary 1.

THEOREM 26. *A coset $(a)_M$ of M has a unique opposite coset of M, namely M/a.*

Proof. Choose a' such that $a'a \approx M$. Then $(a')_M$ is opposite to $(a)_M$ by definition.

To prove uniqueness, we suppose $(b)_M$ opposite to $(a)_M$ and show $(b)_M = M/a$. Suppose $x \subset (b)_M$. By Theorem 25, $xa \approx M$, so that $x \approx M/a$. Thus $(b)_M \subset M/a$. Conversely suppose $x \approx M/a$. Then $xa \approx M$. By Theorem 25, $ab \approx M$. We eliminate a between the last two relations and solve for x:

$$a \approx M/x, \qquad a \approx M/b,$$
$$M/b \approx M/x,$$
$$Mx \approx Mb \qquad \text{(Transposition Law)},$$
$$x \approx Mb/M = (b)_M \qquad \text{(Th. 23)}.$$

Thus $(b)_M = M/a$ and the theorem is proved.

COROLLARY. *Let M be a nonvoid linear set. Then any coset A of M is expressible in the form M/b; conversely M/b always is a coset of M.*

Proof. Let $(b)_M$ be opposite to A. Then A is opposite to $(b)_M$ and is M/b by the theorem. Conversely let M/b be given. By the theorem, M/b is a coset, namely that opposite to $(b)_M$.

This with Theorem 23 gives us two formulas for a coset, namely $(a)_M = aM/M = M/b$. The first, although more complex, expresses $(a)_M$ directly in terms of its element a, the second is simpler but involves the choice of an element b in the coset opposite to $(a)_M$.

As a simple illustration of our theory of congruence modulo a linear set, let G be a Euclidean 3-space and M a fixed point of G. From our observation above in the paragraph following the definition of congruence of elements, $x \equiv a \pmod{M}$ is equivalent to: x is in M (that is, in this case $x = M$) or x and a are on the same side of M. The latter holds if and only if x is in the half-line or ray $\overrightarrow{Ma}$ emanating from M which contains a. Thus the cosets $(a)_M$ are the rays of G which emanate from M, together with point M itself as a sort of degenerate or null ray.* Consequently the decomposition of G determined by the relation congruence mod M corresponds to the important geometrical fact that there is in G a unique ray from M containing each point of G distinct from M. The geometrical significance of congruence of *sets* is now easily given. For simplicity suppose A, $B \supset M$.

* Compare the notion of null vector in the geometrical theory of vectors.

Then $A \equiv B(\text{mod } M)$ holds, if and only if each ray from M which passes through a point of A also passes through a point of B, and vice versa (Fig. 15). That is, $A \equiv B(\text{mod } M)$ if and only if the cones with vertex M and "bases" A, B are identical.

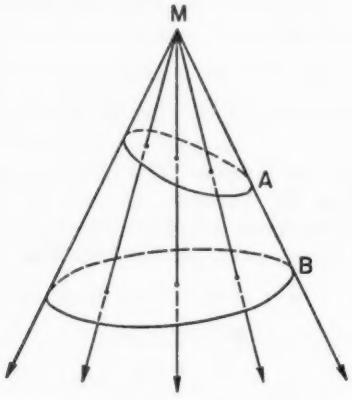


FIG. 15

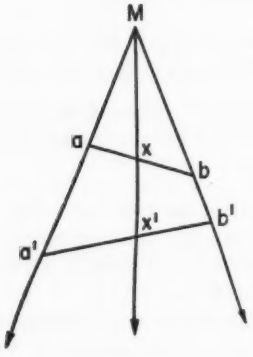


FIG. 16

Furthermore consider the principle: $a \equiv a'(\text{mod } M)$, $b \equiv b'(\text{mod } M)$ imply $ab \equiv a'b'(\text{mod } M)$. As a typical (that is nondegenerate) case suppose a, b, M are noncollinear. Then the rays $\overrightarrow{Ma}$, $\overrightarrow{Mb}$ are distinct and form an angle, namely $\measuredangle aMb$. Moreover a', b' are in the respective sides $\overrightarrow{Ma}$, $\overrightarrow{Mb}$ of the angle. What is the significance of $ab \equiv a'b'(\text{mod } M)$? By our observation above on congruence of sets it asserts that each point x of segment ab (Fig. 16) lies in the same ray from M as some point x' of segment $a'b'$ and vice versa. It implies that any ray from M, the vertex of the angle, which meets one segment joining the sides of the

angle, meets all such segments. This is a very subtle and important property of angles which is intimately related to the notion of angle interior ([4], p. 70, Th. 7). Finally note that opposite cosets will appear as opposite rays from M. You may find it interesting to interpret our other results geometrically, in particular Theorems 21, 23, 25, 26.

Corresponding results hold when M is a linear set in a Euclidean space G of arbitrary dimension. For example, if M is a line and G a 3-space, the cosets are the half-planes of G with edge M together with M as a "degenerate" half-plane, and the theory is applicable to dihedral angles with edge M. Similarly if M is a plane in 4-space G, the nondegenerate cosets are half-3-spaces bounded by M and we have an application to hyperangles formed by two such cosets.

11. Join systems compared with abelian groups. We digress at this point to compare our theory of join systems with abelian group theory. This section is intended to give a deeper insight into the theory of join systems but is actually dispensable for its further development.

Let $(G', \cdot)$ be an abelian group multiplicatively expressed. Then the operations join and extension in a join system $(G, \cdot)$ correspond to multiplication and division in group $(G', \cdot)$, and we can translate the concepts and theorems of our theory into corresponding concepts and statements of abelian group theory. Of course this translation process need not yield important concepts or valid theorems—however I think you will be surprised, at first view, by the frequency with which this does happen.

First, corresponding to the notion of a convex set in G we have the idea of a set closed under multiplication in G'. Linear sets which are nonvoid correspond to nonvoid sets closed under multiplication *and* division—that is to *subgroups* of group G'. The notion of linear closure of set S or linear set generated by S corresponds to the important algebraic idea of subgroup generated by set S ([7], p. 30).

Continuing, we translate into group theory the notion of congruence. This yields $x \equiv y \pmod{M}$, where M is a subgroup of G', in the sense that xM and yM have a common element. Thus

$$xm_1 = ym_2, \qquad x = ym_2m_1^{-1},$$

where $m_1, m_2 \in M$. Hence $x = ym$, $m \in M$. This is the familiar definition of $x \equiv y \pmod{M}$ in group theory ([7], p. 37). Further, our formula for the coset of M determined by a, is in group G'

$$aM/M = a(M/M) = aM,$$

which is the familiar form of a coset in abelian group theory. Thus cosets in the theory of join systems have a more than nominal relation to cosets in abelian groups.

It is interesting that opposite cosets correspond in G' to cosets determined by mutually inverse elements, a and a^{-1}. To show this let A, B be "opposite"

cosets in G'—that is there exist a, b such that $a \in A$, $b \in B$ and $ab \approx M$. We have $ab = m \in M$ and $b = a^{-1}m$. Then $A = aM$ and

$$B = bM = a^{-1}mM = a^{-1}M,$$

justifying our statement. Further note that B, the coset "opposite" to A, is expressible as M/a which is precisely our formula (Th. 26) for the opposite of $(a)_M$.

Many of our theorems hold literally when translated for abelian groups, and practically all have analogues in abelian group theory. In particular the results of the last section go over with the exception of the corollaries to Theorems 23 and 25. The former does not carry over since in a group the only multiplicatively closed coset of M is M itself; counterexamples to the latter are easily found.

We exhibit the basic analogies between the two systems in a table:

Join System $(G, \cdot)$	Abelian Group $(G', \cdot)$
convex set	multiplicatively closed set
linear set	subgroup
linear closure of S	subgroup generated by S
$x \equiv y (\mathrm{mod}\ M)$ is $xM \approx yM$	$x \equiv y (\mathrm{mod}\ M)$ is $x \in yM$
coset (half-space)	coset
coset $(a)_M = aM/M$	coset $(a)_M = aM$
M is a coset $(M = (m)_M$ for $m \subset M)$	M is a coset $(M = (i)_M$, where i is the identity of $G)$
opposite of $(a)_M$ is M/a	"inverse" of $(a)_M$ is $a^{-1}M$

Is it an accident that the two theories are so similar? The answer—as you probably realize by now—is no. Turning back to our postulates [(J1), · · · , (J6) of Sec. 3] we observe that they involve no assumption concerning the number of elements in the join $a \cdot b$, except for the requirement in (J1) that it be nonvoid. Thus $a \cdot b$ may consist of a single element. Suppose then that $a \cdot b$ does consist of a single element for each pair a, b, that is $a \cdot b = (c)$. In accordance with our convention for identifying (c) and c we may write $a \cdot b = c$ and the operation $\cdot$ becomes a single-valued binary operation of the familiar type. In this case (J1), · · · , (J4) reduce to the familiar postulates for an abelian group.

Now let $(G', \cdot)$ be any abelian group. Then $(G', \cdot)$ satisfies (J1), · · · , (J4). Further, a/b must consist of a single element, so that (J5) reduces to the familiar proportion property, $a/b = c/d$ *implies* $ad = bc$, and is valid. (J6) will not in general be valid, since only the identity of G' will satisfy $aa = a = a/a$. In any case we have an unforseen but significant relation between classical geometry and modern algebra: Each theorem in the theory of join systems, whose proof is independent of (J6), is valid for abelian groups.* Our observation above (preceding the table), that the corollaries to Theorems 23 and 25 did not have valid analogues in abelian group theory, now falls into place: The proofs of these corollaries actually do depend on (J6).

* A chauvinistic geometer might be tempted to say that certain portions of modern algebra are merely special cases of a suitably formulated geometric theory.

Join systems are examples of a type of generalized group with many-valued composition called a multigroup. More precisely a multigroup is a system closed under an associative many-valued operation $\cdot$ (that is, it satisfies (J1) and (J3)), which contains elements x, y satisfying the relations, $a \cdot x \supset b$, $y \cdot a \supset b$ for a, b in the system (see [3], pp. 706, 707). Postulates (J1), $\cdots$, (J4) characterize commutative multigroups, which in view of our discussion include commutative or abelian groups.

Theories of multigroups, algebraically motivated, have been given by Ore and his students, among other mathematicians.* These do not seem to have much relevance to geometry. This paper is an outgrowth of my studies of types of multigroups suggested by classical geometries ([9], [10], [12]).

We have now completed the basic presentation of our approach to classical geometry. Join system, our central idea, has its roots in Euclid's operations of joining points and extending segments, and seems to provide an appropriate frame for the geometric study of the important contemporary ideas of convex sets, linear sets and half-spaces.

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* For a list of references on multigroups see R. H. Bruck, A Survey of Binary Systems, Berlin, 1958.

II. FURTHER DEVELOPMENT OF THE THEORY OF JOIN SYSTEMS

In this part we probe more deeply into the theory of join systems and impose additional restrictions on them. The material is subtler and more difficult than that in Part I and some proofs are omitted.

12. The cosets form a geometry. Following a theme familiar in modern mathematics, especially so in algebra, we form a new mathematical system of the cosets $(a)_M$ by constructing a join operation applicable to these cosets. The general reference for this material is [10], Sec. 7.

Let M be a nonvoid linear set in G and let $G:M$ denote the set of cosets $(a)_M$ for $a \subset G$. ($G:M$ may be read G reduced modulo M, or G over M.) We define a join operation in $G:M$ rather naturally by taking $(a)_M \cdot (b)_M$ to be the set of cosets $(x)_M$ for which $x \subset ab$. That is, we form the join ab of representative elements a, b of the given cosets, and then take the set of all cosets of M determined by the elements of ab. It is not hard to see that the result is independent of the choice of the representative elements a, b. For if $(a)_M = (a')_M$ and $(b)_M = (b')_M$ then $a \equiv a' \pmod{M}$ and $b \equiv b' \pmod{M}$ so that $ab \equiv a'b' \pmod{M}$; thus the elements of ab determine the same set of cosets as those of $a'b'$, and $(a)_M \cdot (b)_M = (a')_M \cdot (b')_M$. This has important geometric content which is easily elicited (see the illustration of the theory of congruence at the end of Sec. 10 including Fig. 16).

$G:M$ with join or product as defined is called the *factor system* of G with respect to M * and bears many analogies to its parent system G. In studying $G:M$ we employ $\mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots$ to represent its elements (that is, cosets of M) and adopt the same set-theoretic conventions concerning the use of $\subset$, $\supset$, $\approx$, etc., as in G. We define the inverse operation $\mathbf{a}/\mathbf{b}$ and extend $\cdot$ and $/$ to sets just as in G. In view of this, certain of the results of Sections 4, 5 on join systems, which depend only on the definitions, automatically hold for $G:M$. These include Theorems 1 and 4 (the monotonic laws for join and extension), Theorem 5, and its corollary, *$a \approx bc$ if and only if $a/b \approx c$.*

The basic similarities of $G:M$ and G are given in

THEOREM 27. *$G:M$ satisfies* (J1), $\cdots$, (J5) *and the first idempotent law of* (J6).

Proof. (J1), (J2) are easily seen to hold in $G:M$.

To establish (J3), the associative law, suppose $\mathbf{x} \subset \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$. We reduce this to a corresponding relation in G in order to apply associativity in G. By definition of product of sets

* See [1], p. 154 for the analogous concept in group theory. We adopt the symbolism $G:M$ rather than G/M since the latter has already been employed to denote quotient of sets.

(12.1) $$\mathbf{x} \subset \mathbf{a}\cdot\mathbf{y}, \qquad \mathbf{y} \subset \mathbf{b}\cdot\mathbf{c}$$

holds for some $\mathbf{y}$. Let us examine the first relation in (12.1). Let $\mathbf{a}=(a)_M$, $\mathbf{y}=(y)_M$. Then by definition of $\cdot$, $\mathbf{a}\cdot\mathbf{y}$ or $(a)_M\cdot(y)_M$ is the set of cosets $(x)_M$ for which $x\subset ay$. Thus $\mathbf{x}$ is one of these cosets—that is, $\mathbf{x}=(x)_M$ for some $x\subset ay$. Similarly letting $\mathbf{b}=(b)_M$, $\mathbf{c}=(c)_M$, the second relation in (12.1) yields $\mathbf{y}=(y)_M$ for some $y\subset bc$.

We operate now in G. The relations $x\subset ay$, $y\subset bc$ imply $x\subset a(bc)=(ab)c$. This yields

(12.2) $$x \subset zc, \qquad z \subset ab$$

for some z.

What does (12.2) signify in $G:M$? It says, in view of the definition of $\cdot$,

$$(x)_M \subset (z)_M\cdot(c)_M, \qquad (z)_M \subset (a)_M\cdot(b)_M.$$

Letting $\mathbf{z}=(z)_M$, we have $\mathbf{x}\subset\mathbf{z}\cdot\mathbf{c}$, $\mathbf{z}\subset\mathbf{a}\cdot\mathbf{b}$, so that $\mathbf{x}\subset(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$. Thus

$$\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}) \subset (\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}.$$

The reverse inclusion may be proved similarly to complete the verification of (J3).

To verify (J4) show that if in G, $x\subset a/b$, then in $G:M$, $(x)_M\subset(a)_M/(b)_M$.

To verify (J5) suppose $\mathbf{a}/\mathbf{b}\approx\mathbf{c}/\mathbf{d}$. Then for some $\mathbf{x}$, $\mathbf{x}\approx\mathbf{a}/\mathbf{b}$, $\mathbf{x}\approx\mathbf{c}/\mathbf{d}$, so that $\mathbf{b}\cdot\mathbf{x}\approx\mathbf{a}$, $\mathbf{d}\cdot\mathbf{x}\approx\mathbf{c}$. Let $\mathbf{x}=(x)_M$, $\mathbf{b}=(b)_M$, $\mathbf{d}=(d)_M$. Then

$$(b)_M\cdot(x)_M \supset \mathbf{a}, \qquad (d)_M\cdot(x)_M \supset \mathbf{c}$$

so that $\mathbf{a}=(a)_M$, where $bx\supset a$, and $\mathbf{c}=(c)_M$, where $dx\supset c$. Thus $a/b\approx x\approx c/d$ and $ad\approx bc$. This yields $\mathbf{a}\cdot\mathbf{d}\approx\mathbf{b}\cdot\mathbf{c}$ and (J5) holds in $G:M$.

The idempotent law $aa=a$ in G, implies in $G:M$, $(a)_M\cdot(a)_M=(a)_M$ and our proof is complete.

We consider now the differences between $G:M$ and G. The second idempotent law of (J6) does not in general hold in $G:M$. This is a consequence of the existence of an identity element in $G:M$.

DEFINITION. *If* $\mathbf{a}\cdot\mathbf{i}=\mathbf{a}$ *for each* $\mathbf{a}$ *in* $G:M$ *we call* $\mathbf{i}$ *an identity element of* $G:M$.

THEOREM 28. *$G:M$ has a unique identity element, namely M.*

Proof. By Theorem 21 we have for arbitrary a in G, $am\equiv a(\mathrm{mod}\ M)$, where m is chosen to satisfy $m\subset M$. This is seen to yield in $G:M$

$$(a)_M\cdot(m)_M = (a)_M$$

and $(m)_M$ is an identity in $G:M$. By Theorem 24, $(m)_M=M$. Uniqueness is easily proved.

COROLLARY. *$G:M$ satisfies the idempotent law* $\mathbf{a}/\mathbf{a}=\mathbf{a}$ *only in the trivial case where it consists of the single element M.*

Proof. $\mathbf{a} \cdot M = \mathbf{a}$ implies $\mathbf{a}/\mathbf{a} \supset M$.

Since $G:M$ has an identity we naturally wonder whether elements have inverses. We adopt the

DEFINITION. *In* $G:M$ *if* $\mathbf{a} \cdot \mathbf{x} \supset M$ *we say* $\mathbf{x}$ *is an* ***inverse*** *of* $\mathbf{a}$, *or* $\mathbf{a}$ *and* $\mathbf{x}$ *are* ***inverse elements.***

This definition is reasonable in the present context, since the multiplication is many-valued—the requirement $\mathbf{a} \cdot \mathbf{x} = M$, suggested by conventional algebra, would be unduly restrictive.

In order to analyze the notion, suppose $(a')_M$ is an inverse of $(a)_M$. We have

$$(a)_M \cdot (a')_M \supset M. \tag{12.3}$$

Thus $M = (m)_M$ for some $m \subset aa'$. Clearly $m \subset M$ so that $aa' \approx M$. Consequently $(a)_M$ and $(a')_M$ are *opposite cosets* in the sense of Section 10 (Definition following Th. 24). Conversely if $(a)_M$ and $(a')_M$ are opposite then $aa' \approx M$ which implies (12.3). Thus the geometrically motivated notion of opposite cosets is equivalent to the algebraic notion of inverse elements in $G:M$.

This yields immediate information about $G:M$. Restating Theorem 26 on opposite cosets we have

THEOREM 29. *Each element of* $G:M$ *has a unique inverse.*

A deeper result ([10], p. 362, Th. 6) which we state without proof is

THEOREM 30. *In* $G:M$, $\mathbf{a}/\mathbf{b} = \mathbf{a} \cdot \mathbf{b}'$, *where* $\mathbf{b}'$ *is the inverse of* $\mathbf{b}$.

It seems remarkable that in $G:M$ "division" is reducible to multiplication as in school algebra. Thus in several ways $G:M$ is a simpler and algebraically more regular system than G.*

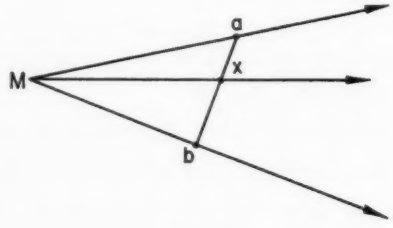

FIG. 17

To illustrate the geometric content of our discussion consider the example at the end of Section 10 in which G is a Euclidean 3-space and M a point. Then $G:M$, the family of cosets of M, consists of the rays of G from M, together with the "null ray" M. The join of the rays $(a)_M$, $(b)_M$ in $G:M$ (Fig. 17) consists of all

* See [12] for a study of multigroups of the type of $G:M$ and a relation to spherical geometries.

rays $(x)_M$ for which $x \subset ab$—that is, it is the set of rays from M which intersect the join ab. So in the typical case where $(a)_M$ and $(b)_M$ form a nonstraight angle (that is, a, b, M are noncollinear), their join $(a)_M \cdot (b)_M$ is actually the set of rays "between" $(a)_M$ and $(b)_M$ or equivalently the set of rays from M which are inside $\measuredangle aMb$.* The identity property asserts that if M is joined to point a, all points of their join determine the same ray (from M) as a. The inverse of any element of $G:M$, other than M, is merely its opposite ray. In fact if $(a)_M$, $(a')_M$ are opposite rays, $(a)_M \cdot (a')_M = ((a)_M, (a')_M, M)$. To summarize: In the given instance, $G:M$ is essentially the natural or intrinsic geometry of the family of rays from M, in which the notion of join (or of betweenness of rays) is that naturally induced by the join concept in G.

We can picture the given system $G:M$ more graphically by passing a sphere S centered at M, and projecting each ray from M into the point where it pierces S (Fig. 18). Then if a, b are points of S in general position (that is are distinct and not opposite to each other), the rays of the join $(a)_M \cdot (b)_M$ will pierce S in

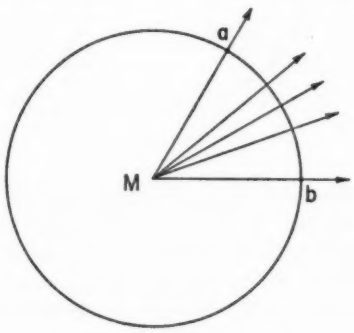

FIG. 18

the points of $\overset{\frown}{ab}$, the (open) minor arc of a great circle joining a and b. Thus our geometry $G:M$ is representable essentially as the natural geometry of a sphere in which the join of two points (in general position) is the minor arc of a great circle joining them.

To characterize more precisely this "spherical" representation of $G:M$, observe first that there is no point of S that represents the "null ray" M of $G:M$. To remedy this we adjoin point M, as a sort of ideal point, to the points of S forming a set S'. Then we define the join $a \cdot b$ for a, b in S' as follows:

(1) If a, b are points of S in general position, $a \cdot b$ is the minor arc of a great circle joining a and b; (2) if a, b are opposite points of S, $a \cdot b = (a, b, M)$; (3) $a \cdot a = a$; (4) $a \cdot M = M \cdot a = a$. This converts S' into what might be termed a "spherical join system" which is a representation of (or is isomorphic to) $G:M$.

* See [2], p. 163.

In this system S', point M is the identity element, and the inverse of any point of S is the opposite point of S. To illustrate the significance of the "quotient" a/b in system S', let a, b be points of S in general position. By definition a/b is the set of all elements x of S' such that the join bx contains a. All such x constitute the minor arc of a great circle $\widehat{ab'}$, where b' is the point of S opposite to b (Fig. 19). Thus $a/b=\widehat{ab'}$. Further we have $a/b=\widehat{ab'}=a\cdot b'$, giving a simple geometric interpretation for Theorem 30.

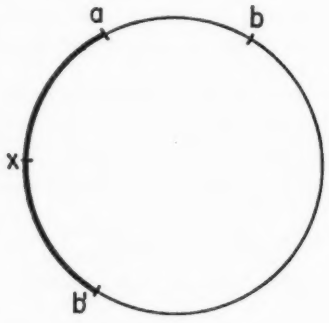

FIG. 19

These ideas are applicable if M is a nonvoid linear set in a Euclidean geometry G of arbitrary dimension—in this case $G:M$ is essentially the natural geometry of the family of half-spaces bounded by M, and also is representable as a "spherical system" ([10], p. 369, Theorem 1).*

We conclude this section with a simple property of factor systems which—as we shall see later—helps to illumine the concept of separation. As in group theory we define the *order* of $G:M$ to be its cardinal number. We call $G:M$ *trivial* if $G=M$, that is if it has order 1.

THEOREM 31. *A nontrivial factor system $G:M$ has order $\geqq 3$.*

Proof. Let $a\subset G$, $a\not\subset M$. Then $G:M$ contains the cosets M, $(a)_M$, and $(a')_M$ the opposite of $(a)_M$. Suppose two of these coincide. Then, since M is the only element of $G:M$ which is its own opposite (Theorem 25 Corollary 2), all three must coincide. Since $a\not\subset M$ this is impossible, and the theorem must hold.

13. Factor systems and homomorphisms. In this section we introduce the idea of homomorphism of a join system, relate it to the notion factor system, and indicate deeper analogies with group theory.

Let M be a given nonvoid linear set of join system G. Naturally we are interested in the mapping T, $x\rightarrow(x)_M$, which assigns to each element x of G the coset of M it determines. T maps G onto $G:M$. We have $a\rightarrow(a)_M$, $b\rightarrow(b)_M$. What can

* See [12] for a "join-theoretic" treatment of spherical geometries.

we say of the image of x, if $x \subset ab$? By definition of join for cosets (Sec. 12), we have $(x)_M \subset (a)_M \cdot (b)_M$. Conversely if $\mathbf{x} \subset (a)_M \cdot (b)_M$, we see that $\mathbf{x} = (x)_M$ for some $x \subset ab$. Thus T maps ab *onto* $(a)_M \cdot (b)_M$ or $T(ab) = T(a) \cdot T(b)$, and we say T maps joins onto joins or *preserves* joins. Note that T is not in general 1-1. Consider our perennial example: M is a point of Euclidean 3-space G. Then essentially T maps G in a "natural" manner onto the geometry of rays from M, or in effect T maps G onto an associated "spherical system."

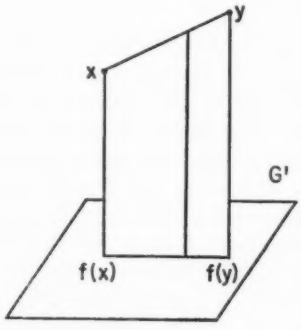

FIG. 20

These considerations call to mind the notion homomorphism of modern algebra.* For convenience we formulate the homomorphism concept for a general type of combinatory system not necessarily a join system. Consider a "combinatory system" $(S, *)$ consisting of a set S and a 2-term operation $*$ which associates to each pair a, b of elements of S a subset of S denoted $a * b$. Let $(S, *)$, $(S', *)$ be two such systems, where for convenience the same symbol is used to denote their operations. Suppose that there exists a mapping f of S on S' satisfying $f(x * y) = f(x) * f(y)$. Then we call f a *homomorphism* of $(S, *)$ on $(S', *)$ and say $(S, *)$ is *homomorphic* to $(S', *)$ or simply S is *homomorphic* to S'. If f is 1-1 we say S is *isomorphic* to S' and write $S \cong S'$.

We observe immediately that $x \to (x)_M$ is a homomorphism of join system G on $G:M$. You might expect, as in group theory ([1], p. 154, Th. 26; [7], Ch. I, Sec. 16), that every homomorphism of a join system G is associated with a factor system $G:M$. This is not so. The correct result is: Let f be a homomorphism of join system G on a combinatory system G', which has an identity i in the sense that $x * i = i * x = x$ for x in G'. Then G' is isomorphic to $G:M$, where M is the *kernel* of f, that is the set of elements mapped on i by f ([**10**], p. 366, Corol. 2). Thus the study of homomorphisms here is essentially more complex than in group theory, since there exist homomorphisms of a join system on a second join system which has no identity. Actually there is a very familiar example—an

* See, for example, [1], pp. 150–151; [7], pp. 41–43.

orthogonal projection f of Euclidean 3-space on plane G' (Fig. 20) clearly is a homomorphism of G on G' considered as join systems.

A sizable portion of the theory of groups involving factor groups and homomorphism is valid in our theory with suitable modification, including the Jordan-Holder theorem and various isomorphism theorems. An important example, rich in geometrical content, is

THEOREM 32 (ISOMORPHISM THEOREM). *Let A, B be linear sets in G; $A \cap B \neq \emptyset$. Then $\{A, B\} : A \cong B : (A \cap B)$.*

Here we are considering $\{A, B\}$ as a join system relative to the join operation in G (see last paragraph, Sec. 8) and A as a linear set in $\{A, B\}$; similarly for B and $A \cap B$. See [10], page 366, Theorem 3 for the proof; [7], page 136 for the corresponding result in group theory in generalized form.

Geometric content: In Euclidean 3-space G let A be a line, B a plane such that $A \cap B$ is a point (Fig. 21). Then $\{A, B\} = G$ and $\{A, B\} : A$ is essentially the geometry of the half-planes in G with edge A. Similarly $B : (A \cap B)$ is essentially the geometry of the rays in B from $A \cap B$. Clearly the two systems are isomorphic—essentially each is the natural geometry of a circle. Moreover we can establish the isomorphism by a simple geometric procedure, namely: Associate with each element of $\{A, B\} : A$ its intersection with plane B.

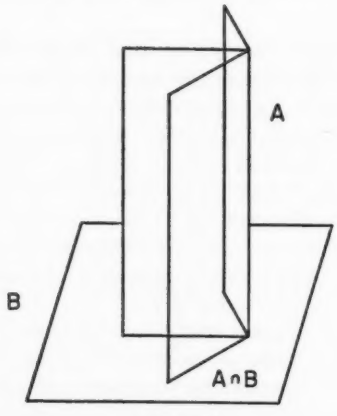

FIG. 21

14. Incidence relations and dimensionality. Our treatment of linear sets in an arbitrary join system (Secs. 8, 9) was quite general. It did not cover familiar incidence relations for example, "three noncollinear points belong to a unique plane," or intersection properties like "in a 3-space if two distinct planes intersect, their intersection is a line." In fact we have not defined "3-space" or even "plane." Nor have we introduced a notion of dimension to classify linear sets and organize them into a hierarchy. These related matters we consider now.

We can not, on the basis of (J1), $\cdots$, (J6), derive the familiar incidence properties of Euclidean geometry—in fact the property "two distinct points belong to a unique line" is independent of (J1), $\cdots$, (J6). This is justified by an example in the Appendix (Sec. E) of a join system which does not satisfy the stated property. Consequently to obtain the Euclidean theory of incidence it is necessary to postulate this property or an equivalent. It is interesting that the property is sufficient. Specifically we can assume it in the following weakened form:

(E) (EXCHANGE POSTULATE). *If* $b \subset \{a_1, a_2\}$, $b \neq a_2$ *then* $\{a_1, a_2\} = \{b, a_2\}$.

This says in effect that if b belongs to line a_1a_2, the pairs a_1, a_2 and b, a_2 determine the same line. Although (*E*) has not the simple algebraic character of (J1), $\cdots$, (J6), it is algebraic in nature and may be described as an "exchange" principle, since it permits us to exchange a_1 for b in $\{a_1, a_2\}$ without affecting the result.

We develop an incidence theory for join systems satisfying (E). First (E) is generalized to n terms:

THEOREM 33 (EXCHANGE LEMMA). *Suppose that* $b \subset \{a_1, \cdots, a_n\}$, $b \not\subset \{a_2, \cdots, a_n\}$. *Then* $\{a_1, \cdots, a_n\} = \{b, a_2, \cdots, a_n\}$.

Proof. The relation $b \subset \{a_1, \cdots, a_n\}$ implies, by definition of linear closure of a set, $\{b, a_2, \cdots, a_n\} \subset \{a_1, \cdots, a_n\}$. The reverse inclusion is immediate by the same argument, once we have established

$$a_1 \subset \{b, a_2, \cdots, a_n\}. \tag{14.1}$$

The proof of (14.1) is longer. We have

$$\begin{aligned} b &\subset \{a_1, \cdots, a_n\} \\ &\subset \{\{a_1, a_2\}, \{a_2, \cdots, a_n\}\} \\ &= \{a_1, a_2\}/\{a_2, \cdots, a_n\} \qquad \text{(Theorem 18).} \end{aligned}$$

This yields

$$b \subset p/q \tag{14.2}$$

where p, q satisfy

$$p \subset \{a_1, a_2\}, \tag{14.3}$$

$$q \subset \{a_2, \cdots, a_n\}. \tag{14.4}$$

Relation (14.2) implies

$$p \subset bq. \tag{14.5}$$

In order to apply the Exchange Postulate to (14.3) we need $p \neq a_2$. This is valid, since $p = a_2$ implies by (14.2), (14.4), $b \subset \{a_2, \cdots, a_n\}$, contrary to hypothesis.

Hence

$$\begin{aligned} a_1 &\subset \{a_1, a_2\} \\ &= \{p, a_2\} && \text{(Exchange Postulate)} \\ &\subset \{b, q, a_2\} && (14.5) \\ &\subset \{b, a_2, \cdots, a_n\} && (14.4). \end{aligned}$$

Thus (14.1) holds and the proof is complete.

In order to deal uniformly with n points in "general position," for example two distinct points or three noncollinear points, we introduce the idea of linear independence.

DEFINITION. *We say* $a_1, \cdots, a_n$ *are* ***linearly independent*** *or simply* ***independent*** *if they are never contained in a linear set generated by* $n-1$ *elements—that is, if the statement* $\{x_1, \cdots, x_{n-1}\} \supset a_1, \cdots, a_n$ *is false for every choice of* $x_1, \cdots, x_{n-1}$.

If $n=1$, $\{x_1, \cdots, x_{n-1}\}$ is taken to be $\{\varnothing\}$ which is $\varnothing$. Thus a single element a_1 is independent. Note that a_1, a_2 are independent if and only if $a_1 \neq a_2$.

We can show without difficulty: (1) if $a_1, \cdots, a_n$ are independent they are distinct; (2) that any nonvoid subset of an independent set is also independent.

The following theorem and corollary prepare for the notion of dimension.

THEOREM 34. *If* $\{x_1, \cdots, x_m\} \supset a_1, \cdots, a_n$ *and* $a_1, \cdots, a_n$ *are independent then* $m \geqq n$.

Proof. By definition of independence $m \neq n-1$. Suppose $m < n-1$. Then we can insert redundantly a particular one of the x's, say x_1, in the expression $\{x_1, \cdots, x_m\}$ a sufficient number of times, so that it appears as a linear set generated by $n-1$ elements. This is impossible by definition of independence and we infer $m \geqq n-1$.

COROLLARY. *Let* $\{a_1, \cdots, a_m\} = \{b_1, \cdots, b_n\}$, *where* $a_1, \cdots, a_m$ *are independent and* $b_1, \cdots, b_n$ *are independent. Then* $m = n$.

Proof. $\{a_1, \cdots, a_m\} \supset b_1, \cdots, b_n$ so that $m \geqq n$ by the theorem. Symmetrically $n \geqq m$.

When determining a linear set it is desirable, if possible, to choose a set of generators which is independent. This suggests the

DEFINITION. *If* $a_1, \cdots, a_n$ *is an independent set of generators of linear set* A, *that is, if* $A = \{a_1, \cdots, a_n\}$ *and* $a_1, \cdots, a_n$ *are independent, we call* $a_1, \cdots, a_n$ *a* ***basis*** *of* A.

Restating the last corollary in terms of the idea of basis, we have an important uniqueness result:

THEOREM 35. *All bases of linear set A have the same cardinal number.*

The unique cardinal number associated with linear set A indicates its relative complexity of structure and suggests the idea dimension:

DEFINITION. *If linear set A has a basis, the unique cardinal number of each of its bases is called its* ***rank or dimension*** *and is denoted by $d(A)$. We define $d(\varnothing)=0$.*

Dimension as defined exceeds by unity the conventional definition—for example, the dimension of a Euclidean line is 2, of a Euclidean plane is 3, etc. In the present context this is quite natural and very convenient. In our join theory the dimension of an element a is 1, that of a line ($\{a, b\}$, $a \neq b$) is 2, and we *define* a plane to be a linear set A such that $d(A)=3$.

Familiar determination properties of Euclidean lines and planes generalize to

THEOREM 36 (EXCHANGE THEOREM). *Suppose $A=\{a_1, \cdots, a_n\}$. Then any n independent elements of A form a basis of A.**

Proof. Suppose $\{a_1, \cdots, a_n\} \supset (b_1, \cdots, b_n)$ and the b's are independent. We employ the Exchange Lemma (Th. 33) to exchange a's for b's, one by one, in $\{a_1, \cdots, a_n\}$. Since $b_1, \cdots, b_n$ are independent, $\{a_2, \cdots, a_n\} \not\supset (b_1, \cdots, b_n)$ and one of the b's, for simplicity say b_1, is not in $\{a_2, \cdots, a_n\}$. By the Exchange Lemma we have

$$\{a_1, \cdots, a_n\} = \{b_1, a_2, \cdots, a_n\} = \{a_2, \cdots, a_n, b_1\}. \tag{14.6}$$

We apply a similar argument to $\{a_2, \cdots, a_n, b_1\}$. Clearly $\{a_2, \cdots, a_n, b_1\} \supset b_2, \cdots, b_n$. Further $\{a_3, \cdots, a_n, b_1\} \not\supset (b_2, \cdots, b_n)$. Hence one of the indicated b's, say b_2, is not in $\{a_3, \cdots, a_n, b_1\}$. By the Exchange Lemma

$$\{a_2, \cdots, a_n, b_1\} = \{b_2, a_3, \cdots, a_n, b_1\} = \{a_3, \cdots, a_n, b_1, b_2\},$$

and (14.6) implies $\{a_1, \cdots, a_n\} = \{a_3, \cdots, a_n, b_1, b_2\}$. Continuing to exchange a's for b's in this way we get eventually $\{a_1, \cdots, a_n\} = \{b_1, \cdots, b_n\}$ and the theorem holds.

The exchange theorem has many important consequences.

COROLLARY 1. *There is a unique linear set of dimension n containing a given set of n independent elements.*

Proof. Suppose $b_1, \cdots, b_n$ are independent. Then $\{b_1, \cdots, b_n\}$ satisfies the existence condition. Uniqueness follows, since if $d(A)=n$ and $A \supset b_1, \cdots, b_n$ then $A=\{b_1, \cdots, b_n\}$ by the theorem.

Taking $n=2$ we get the property: *Two distinct points belong to a unique line.*

Further we get a procedure for enlarging independent sets:

COROLLARY 2. *Let $a_1, \cdots, a_n$ be independent and $a_{n+1} \not\subset \{a_1, \cdots, a_n\}$. Then $a_1, \cdots, a_n, a_{n+1}$ are independent.*

* Similar properties occur in modern algebra, for example see [1] p. 169, Theorem 5 Corollary.

Proof. Suppose the conclusion false. Then

$$\{x_1, \cdots, x_n\} \supset a_1, \cdots, a_n, a_{n+1}$$

for some $x_1, \cdots, x_n$. Using the theorem,

$$a_{n+1} \subset \{x_1, \cdots, x_n\} = \{a_1, \cdots, a_n\},$$

contrary to hypothesis.

This easily yields an inductive criterion for independence:

COROLLARY 3. *$a_1, \cdots, a_n$ are independent if and only if*

$$a_i \not\subset \{a_1, \cdots, a_{i-1}\}, \qquad 1 \leqq i \leqq n. \tag{14.7}$$

Proof. Suppose (14.7). The case $n=1$ is trivial since a_1 is independent. Assume $n>1$. Let $i=2$; since a_1 is independent, Corollary 2 implies a_1, a_2 are independent. Similarly the latter implies a_1, a_2, a_3 are independent. Continuing in this way we get eventually the independence of $a_1, \cdots, a_n$.

To show the converse, suppose (14.7) fails for a given i. Then

$$\{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\} \supset a_1, \cdots, a_n$$

and $a_1, \cdots, a_n$ are not independent.

Now the following familiar criterion for independence can be derived.

COROLLARY 4. *$a_1, \cdots, a_n$ are independent if and only if*

$$a_i \not\subset \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\}, \qquad 1 \leqq i \leqq n.$$

Corollary 3 also yields an important criterion for a basis:

COROLLARY 5. *A maximal independent subset of linear set A is a basis of A.*

Proof. Suppose $a_1, \cdots, a_n$ is an independent subset of A and no "larger" set has this property. Clearly $A \supset \{a_1, \cdots, a_n\}$. Let $x \subset A$. If $x \not\subset \{a_1, \cdots, a_n\}$ then Corollary 3 implies $a_1, \cdots, a_n, x$ are independent, contrary to hypothesis. Thus $x \subset \{a_1, \cdots, a_n\}$ and $A = \{a_1, \cdots, a_n\}$.

Now we have a criterion that a linear set have a dimension:

COROLLARY 6. *If a linear set has a finite set of generators it has a dimension.*

Proof. $\{a_1, \cdots, a_n\}$ has a maximal independent subset containing at most n elements in view of Theorem 34. Hence by Corollary 5 it has a basis and so a dimension.

Another consequence of Corollary 5 is the following monotonic property of dimension:

COROLLARY 7. *If A, B are linear, $A \subset B$, and B has a dimension, then A has a dimension and $d(A) \leqq d(B)$. Further, $d(A) = d(B)$ only if $A = B$.*

Now we consider intersection of linear sets. Familiar intersection properties

of Euclidean lines and planes are generalized in the following deep-lying theorem which we present without proof.

THEOREM 37 (DIMENSION PRINCIPLE). *Let A, B be linear sets which have dimensions and let $A \cap B \neq \varnothing$. Then*

$$d(\{A, B\}) + d(A \cap B) = d(A) + d(B).^*$$

As an application we prove: In a Euclidean 4-space if two intersecting planes are not contained in a 3-space, their intersection is a point.

Let A, B be the planes and C the 4-space. Clearly $\{A, B\} \subset C$ which implies

$$d(\{A, B\}) \leqq d(C) = 5$$

by Corollary 7 above, the monotonic property of dimension. Further $d(\{A, B\}) > 4$; since otherwise there would be a 3-space containing A and B. Thus $d(\{A, B\}) = 5$ and the Dimension Principle asserts

$$5 + d(A \cap B) = 3 + 3,$$

so that $d(A \cap B) = 1$ and the intersection of A and B is a point.

If $A \cap B = \varnothing$ the Dimension Principle is replaced by the inequality

$$d(\{A, B\}) \leqq d(A) + d(B). \tag{14.8}$$

This suggests in n-dimensional Euclidean geometry a general definition of parallelism and skewness of linear sets: If $A \cap B = \varnothing$ and (14.8) is strictly an inequality we say A and B are *parallel*, otherwise they are *skew*.

Finally we note that a linear set of a join system need not have a dimension as we have defined it—for it need not have a finite set of generators. In other words a join system may have linear sets to which our theory does not apply. However our theory of dimension can be extended to cover the linear sets of any join system. To do this a definition of (linear) independence is framed that is applicable equally to finite and infinite sets and is equivalent to our definition in the finite case (see [10], p. 355, Def. 1). Then it is proved that any linear set A has an independent set of generators; and that all independent sets of generators of A have the same cardinal number. This cardinal number is defined to be the dimension of A. If the dimension of A is a finite cardinal, A is said to be *finite* dimensional, otherwise it is *infinite* dimensional. Thus linear sets which have a dimension in our theory are finite dimensional in the generalized theory and the others are infinite dimensional in it. The dimension of a join system G is defined to be that of G as a linear set. In the Appendix (Sec. C) we give an example of an infinite dimensional join system. See [10], Section 9 for more information on the theory of dimension outlined above, and [8], Sections 3, 4 for its justification.†

* This result is proved in [8] for a class of lattices which covers the present situation.

† For such an unrestricted theory of dimension, in projective geometry see [9]; in the theory of vector spaces see R. Baer, Linear Algebra and Projective Geometry, New York, 1952, Chapter II, or N. Jacobson, Linear Algebra, vol. II, New York, 1953, Chapter IX.

15. The concept of order. We have been dwelling in the realm of higher dimensional linear sets, now we descend to consider the notion of order, which is concerned basically with the lowly linear question of how collinear points are related. Order, one of the most pervasive mathematical concepts, is particularly important in geometry. It is intimately involved not merely in purely linear matters like the relative position of points of a line, but is the basis for the study of separation (for example, a plane in a 3-space *separates* it into two half-spaces), and the idea of interiority (for example, a point is *inside* triangle abc if it is between a and a point which is itself between b and c).

Hilbert in his pioneer work [5] begins with incidence postulates relating the primitive terms point, line, plane. Then he introduces order in the form of a 3-term relation, "betweenness," which satisfies postulates ensuring that each line is a "linearly ordered" system.* Veblen in his classic foundations for Euclidean geometry [13, 14] takes point and a 3-term order (or betweenness) relation as primitive terms and defines segment, ray, line and plane in terms of them.

Order is implicit in our theory of join systems. For in Euclidean geometry we have defined (Sec. 2) the join ab of the distinct points a, b as the segment with endpoints a and b; so that any point of ab is between a and b. We convert the latter property into a definition of betweenness in the abstract theory of join systems:

DEFINITION. *If $x \subset ab$ and $a \neq b$ we say x is* ***between*** *a and b and we write (axb).*

Clearly $x \subset ab$ if and only if (axb) or $x = a = b$. In view of this (J1), $\cdots$, (J6) can be rephrased in terms of betweenness. For example, (J2) is equivalent to the familiar order property, (abc) implies (cba). Many other familiar properties of linear order are deducible from (J1), $\cdots$, (J6). The following theorems, which hold in an arbitrary join system (we are not assuming Postulate (E)), are examples of such properties.

THEOREM 38. *(abc) implies that a, b, c are distinct.*

Proof. By hypothesis $b \subset ac$ and $a \neq c$. Suppose $a = b$. Then $a \approx ac$ so that $a/a \approx c$ and (J6) implies $a = c$. This contradiction implies $a \neq b$. Similarly $b \neq c$.

THEOREM 39. (abc), (acd) imply (abd), (bcd).

Proof. By definition we have

$$b \subset ac, \qquad c \subset ad \tag{15.1}$$

where $a \neq c, d$. We eliminate c from (15.1) so: the first relation implies $b/a \supset c$, which yields with the second relation $b/a \approx ad$ and $b \approx aad = ad$. By definition (abd) holds.

Next we eliminate a from (15.1). We have $b/c \approx a$, $c/d \approx a$ so that $b/c \approx c/d$

* See [6] for characterizations of betweenness.

and $bd \approx c$. By Theorem 38, (abd) implies $b \neq d$ and (bcd) holds by definition.

Similarly we can prove that (abc), (bcd) imply (abd), (acd).

However all familiar properties of linear order do not hold in an arbitrary join system or even in one that satisfies Postulate (E). The most important example is the following

COMPARABILITY PROPERTY. *If p, q, r are distinct and collinear then at least one of (pqr), (qpr), (qrp) holds.*

An independence example for this property which satisfies (J1), $\cdots$, (J6) and (E), called a *partially ordered* join system, is described in the Appendix (Sec. G). Thus to obtain, in a join system, the familiar theory of linear order we shall have to postulate the Comparability Property or an equivalent.

Naturally we prefer to formulate the property in join terminology. Suppose then $x \subset \{a, b\}$ and assume $a \neq b$. Since x, a, b colline, the Comparability Property implies (xab), (axb), (abx) or $x = a$ or $x = b$, and so

$$x \subset a/b \cup ab \cup b/a \cup a \cup b,$$

which implies

$$\{a, b\} \subset a/b \cup ab \cup b/a \cup a \cup b. \tag{15.2}$$

Clearly (15.2) holds if $a = b$. The reverse inclusion to (15.2) has been established in Section 9 as relation (9.1). This suggests the following postulate which is a consequence of the Comparability Property:

(C) (COMPARABILITY POSTULATE). $\{a, b\} = a/b \cup ab \cup b/a \cup a \cup b$.

Since, when $a \neq b$, the right member of (C) is the conventional definition of line ab (Sec. 8, (8.1)) and the left member is the one we have adopted (Sec. 9), it may be good to discuss their connection and the significance of (C). Suppose we had defined line ab in the conventional way in Section 9. Immediately there would arise the awkward possibility that a line need not be a linear set. This is realized in the independence example for Postulate (E) given in the Appendix (Sec. E), where $a/b \cup ab \cup b/a \cup a \cup b$ is not linear and in fact is not even convex.

Adopting the suggested definition would consequently force us to postulate that line ab is a linear set, which implies line $ab = \{a, b\}$, $(a \neq b)$. Thus in effect we would have assumed Postulate (C) in Section 9. This would have restricted prematurely the scope of our theory in Sections 9–13, which is applicable to any join system at all. It would have excluded join systems in which Postulate (E) fails (see Appendix Sec. E), since (E) is a consequence of Postulate (C) (see below, Th. 40 Corol. 2), as well as "partially ordered" join systems in which (E) holds but (C) fails (see Appendix Sec. G).

Finally observe that although we have adopted (C) as a kind of comparability principle, motivated by the obvious importance of comparability in the theory of linear order, it is intrinsically worth studying as a property of join systems.

For in view of relation (9.1), a join system satisfying (C) is of the simplest possible type, since in it the expression for $\{a, b\}$ is as simple as possible.

This concludes our discussion of the significance of (C)—its consequences are developed in the next section.

16. Separation and factor systems. In this section we study separation of linear sets by linear sets, relating the idea to properties of factor systems. We derive a general separation theorem for arbitrary linear sets in a join system that satisfies Postulate (C). As a by-product we find that (C) implies the Exchange Postulate (E) of Section 14.

The classical treatment of linear separation proceeds in piecemeal fashion. First it is shown ([4], p. 51, Th. 8) that a line is separated by any of its points. Then it is proved ([4], p. 68, Th. 2) that a plane is separated by any contained line, on the basis of the famous

POSTULATE OF PASCH. *Suppose line* $L \subset$ *plane* $\{a, b, c\}$ *and* $L \not\supset a, b, c$. *Then* $L \approx ab$ *implies* $L \approx bc$ *or* $L \approx ac$, *but not both* (see Fig. 22).*

Finally Pasch's Postulate is generalized to 3-space ([4], p. 65, Th. 30) and it is shown ([4], p. 85, Th. 25) that any plane separates 3-space.

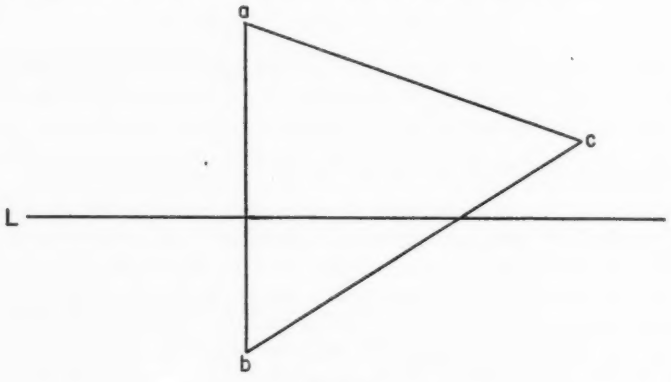

FIG. 22

We give a uniform, dimension-free treatment of the problem without using Pasch's postulate. First we formulate the separation idea in join-theoretic terms. Suppose in a Euclidean geometry, A, B are linear and B separates A; for example, a point separates a containing line or a 3-space separates a containing 4-space. The conventional definition of this may be expressed as follows: There exist nonvoid sets α, β such that (a) $A = \alpha \cup \beta \cup B$; (b) α, β, B are disjoint in pairs; (c) if $x \subset \alpha$, $y \subset \beta$ then $xy \approx B$; (d) if $x, y \subset \alpha$ or $x, y \subset \beta$ then $xy \approx B$ is false. (Fig. 23 illustrates the case where A is a plane and B a line.)

* Compare [4], p. 59, Theorems 16, 16.1.

Let $a \subset \alpha$ (see Fig. 23). It is not difficult to show that $\beta = B/a$, so that β is a coset of B by Theorem 26, Corollary. Similarly α is a coset of B (in fact α and β are opposite cosets by definition (Sec. 10) in view of (c)). Further (a), (b) imply that α, β, B constitute the coset decomposition of linear set A determined by its linear subset B. The essential point is that A decomposes into *three* cosets, which are distinct since the order of $A:B$ is at least 3 by Theorem 31. So our analysis yields this result: $A:B$ has order 3. This suggests the identification of the classical geometric notion "B separates A" with the join-theoretic idea "$A:B$ has order 3."

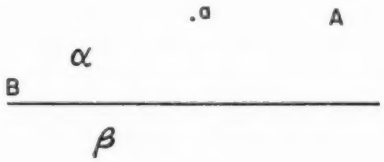

FIG. 23

To test the reasonableness of this suggestion, suppose in a Euclidean geometry A, B are linear sets such that $A:B$ has order 3. Let α, β be the elements of $A:B$ distinct from B. Then A is decomposed into α, β, B and (a), (b) above hold. Further since $A:B$ contains the opposite of each of its elements, and B alone is its own opposite (Th. 25, Corol. 2), the opposite of α must be β. Thus (c) holds by Theorem 25. Finally (d) holds since a coset is convex (Th. 23, Corol.), and B separates A in the conventional sense.

Thus we adopt the

DEFINITION. *If A, B are linear and the order of $A:B$ is 3, we say B* ***separates*** *A.*

In the remainder of this section we consider join systems that satisfy Postulate (C) (Sec. 15). The basic theorem on separation of linear sets is preceded by three lemmas.

LEMMA 1. *$\{a, b\}:b$ has order 3, provided $a \neq b$.*

Proof. Since $\{a, b\} \neq b$, $\{a, b\}:b$ is nontrivial and its order $r \geqq 3$ by Theorem 31. Let $(x)_b$ be any element of $\{a, b\}:b$. We show $(x)_b$ is one of the following cosets, $(a)_b$, its opposite b/a, or b itself. This is trivial if $x = a$ or $x = b$. Suppose $x \neq a, b$. By Postulate (C) one of the following holds:

$$(1)\ x \subset a/b, \qquad (2)\ x \subset ab, \qquad (3)\ x \subset b/a.$$

Suppose (1). Then $xb \approx a$ so that $xb \approx ab$ and $x \approx ab/b = (a)_b$ by Theorem 23. Thus $(x)_b = (a)_b$. Similarly (2) implies $xb \approx ab$, again yielding $(x)_b = (a)_b$. Finally (3) implies $(x)_b = b/a$. Thus $r = 3$ and the lemma holds.

LEMMA 2. *Suppose $A \supset B \supset C$, where A, B, C are linear and $C \neq \varnothing$. Then the order of $A:B$ does not exceed that of $A:C$.*

Proof. By Theorem 23, for $a \subset A$, $(a)_B = aB/B \supset aC/C = (a)_C$. The lemma follows.

LEMMA 3. *If* $A:C = B:C$, *then* $A = B$.

Proof. A and B are the unions of the elements of $A:C$ and $B:C$ respectively.

Now we are prepared to substantiate the general separation theorem for linear spaces. Roughly stated it says if linear space A contains linear space B and is of "next higher rank" then B separates A. How can we clarify the somewhat vague phrase, A is of next higher rank than B, as for example when A is a plane and B a line? We might be tempted to define this by $d(A) = d(B) + 1$, but this imposes the unnecessary restriction that A and B have dimensions (see discussion at end of Sec. 14). The answer is given in the following

DEFINITION. *Let* A, B *be linear sets such that* $A \supset B$, $A \neq B$, *and there exists no linear set* X *"between"* A *and* B, *that is,* $A \supset X \supset B$ *for linear* X *implies* $X = A$ *or* $X = B$. *Then we say* A **covers** B.*

The separation theorem is now expressible in the following simple form:

THEOREM 40 (SEPARATION THEOREM FOR LINEAR SETS). *B separates A if and only if A covers B and* $B \neq \varnothing$.

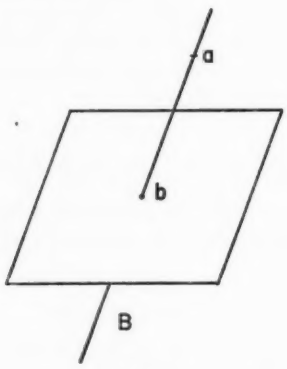

FIG. 24

Proof. Suppose A covers B, $B \neq \varnothing$. Let $a \subset A$, $a \not\subset B$ and $b \subset B$. (Fig. 24 illustrates the case where B is a plane.) We show $A:B$ isomorphic to a factor system of the form $\{a, b\}:B'$. We have

$$A \supset \{B, a\} \supset B, \qquad \{B, a\} \neq B.$$

These imply, by definition of A covers B, $A = \{B, a\} = \{B, \{a, b\}\}$. By the

* This is related to the notions of maximal ideal or maximal subgroup of modern algebra and has been abstracted to lattice theory; see [1], p. 349.

Isomorphism Theorem (Th. 32)

$$A:B = \{B, \{a, b\}\}:B \cong \{a, b\}:(B \cap \{a, b\}) = \{a, b\}:B',$$

where $B' = B \cap \{a, b\}$. We have

$$\{a, b\} \supset B' \supset b, \qquad \{a, b\} \neq B'.$$

Hence, by Lemma 2, the order r of $\{a, b\}:B'$ does not exceed that of $\{a, b\}:b$, which, by Lemma 1, is 3. However $r \geqq 3$ by Theorem 31, so that $r = 3$. Thus the order of $A:B$ must be 3, and B separates A by definition.

Conversely suppose B separates A. Clearly $A \supset B$, $A \neq B$. Suppose $A \supset X \supset B$, $X \neq B$, where X is linear. Then in view of Theorem 23 we have

$$A:B \supset X:B. \tag{16.1}$$

In (16.1) $A:B$ has order 3 by definition of separation, and $X:B$ has order at least 3 by Theorem 31. Hence $A:B$ and $X:B$ both have order 3 and (16.1) implies $A:B = X:B$. By Lemma 3 this yields $X = A$, and A covers B by definition.

Note in the theorem that there is no dimensionality restriction on the join system considered, or on its linear sets A and B; for example, A and B may be infinite dimensional (see discussion at end of Sec. 14). Further, the theorem includes the necessity of the covering condition for separation, which is not considered in the conventional treatment.

COROLLARY 1. *Suppose A covers B, $B \neq \varnothing$, $a \subset A$, $a \not\subset B$, $aa' \approx B$. Then A has a partition of the form $A = B \cup B/a \cup B/a'$.*

Proof. By the theorem, B separates A, so that A has a partition composed of three cosets of B. It is easily shown that B, B/a, B/a' are distinct.

COROLLARY 2. *In a join system Postulate* (C) *implies Postulate* (E).

Proof. Suppose $b \subset \{a_1, a_2\}$, $b \neq a_2$. Then $a_1 \neq a_2$ and by Lemma 1, $\{a_1, a_2\}:a_2$ has order 3. Thus a_2 separates $\{a_1, a_2\}$ and $\{a_1, a_2\}$ covers a_2 by the theorem. But

$$\{a_1, a_2\} \supset \{b, a_2\} \supset a_2, \qquad \{b, a_2\} \neq a_2.$$

Therefore $\{a_1, a_2\} = \{b, a_2\}$ and Postulate (E) is verified.

When we introduced Postulate (C) in Section 15 we motivated it as a *consequence* of the Comparability Property, emphasizing the importance of the latter as an essential property of linear order. Now we are able to complete the discussion of the relation between these properties.

COROLLARY 3. *In a join system Postulate* (C) *implies the Comparabilty Property of Section* 15.

Proof. Suppose p, q, r distinct and collinear. By Corollary 2, Postulate (E) holds and so the results of Section 14 are valid. Thus $\{q, r\}$ is the only line containing q, r (Th. 36, Corol. 1) and $p \subset \{q, r\}$. By Postulate (C)

$$p \subset q/r \cup qr \cup r/q$$

so that (pqr) or (qpr) or (prq) and the Comparability Property follows.

The separation theorem, despite its broad scope, does not immediately justify familiar Euclidean properties such as, a point separates a containing line, or a line separates a containing plane. For the required covering conditions —that a line cover a contained point, or a plane a contained line—have not yet been established. Using Corollary 2 above we take care of this now, in general terms, by relating covering conditions and dimensionality properties.

THEOREM 41. *If A covers B and $d(A)$ exists then $d(A) = d(B) + 1$. Conversely if $A \supset B$ and $d(A) = d(B) + 1$ then A covers B.*

Proof. Observe that by Corollary 2 above, we may apply the dimension theory of Section 14. Suppose A covers B and $d(A)$ exists. If $B = \varnothing$ then $d(A) = 1 = d(B) + 1$. Suppose $B \neq \varnothing$. By Theorem 36, Corollary 7, $d(B)$ exists. Let $B = \{b_1, \cdots, b_n\}$, where $b_1, \cdots, b_n$ are independent. Let $a \subset A$, $a \not\subset B$. Then

$$A \supset \{a, b_1, \cdots, b_n\} \supset B,$$

so that $A = \{a, b_1, \cdots, b_n\}$. The indicated set of generators of A is independent by Theorem 36, Corollary 2. Therefore $d(A) = n + 1 = d(B) + 1$.

Conversely suppose $A \supset B$ and $d(A) = d(B) + 1$. Note $A \neq B$. Suppose $A \supset X \supset B$, where X is linear. By Theorem 36, Corollary 7,

$$d(A) \geqq d(X) \geqq d(B),$$

so that $d(X) = d(A)$ or $d(X) = d(B)$. By Theorem 36, Corollary 7, $X = A$ or $X = B$, and A covers B by definition.

COROLLARY 1. *Suppose $A \supset B \neq \varnothing$, and $d(A) = d(B) + 1$. Then B separates A.*

Proof. A covers B by the theorem and B separates A by Theorem 40.

COROLLARY 2. *In a join system Postulate* (C) *implies the Postulate of Pasch.*

Proof. Suppose line $L \subset$ plane $\{a, b, c\}$, $L \not\ni a, b, c$ and $L \approx ab$. Since

$$d(\{a, b, c\}) = 3 = d(L) + 1,$$

the theorem implies $\{a, b, c\}$ covers L. By Theorem 40, Corollary 1 we have the partition $\{a, b, c\} = L \cup L/a \cup L/b$. Hence $c \subset L/a$ or $c \subset L/b$ but not both, which implies $ac \approx L$ or $bc \approx L$ but not both.

For an extension of the methods developed here to the more complicated problem of the decomposition of a linear set of dimension n effected by an n-simplex, see [10], Section 11. This also contains ([10], Sec. 10) a more sophisticated treatment of the theory of separation of linear sets, based on a deeper analysis of the factor systems involved.

17. Conclusion. This concludes our approach to classical geometry. We have given a join-theoretic basis for the classical theories of incidence and order of Euclidean geometry. Since we have assumed no parallel postulate our treatment is equally valid in hyperbolic (Lobachevskian) geometry. In fact our study of join systems that satisfy Postulate (C) is a dimensionally (and existentially) unrestricted theory of the ordered linear geometries which have been studied in the foundations of geometry under the name descriptive geometries.*

We have approached but not reached classical geometry—much remains to be done. It is not wholly clear which is the best path to follow in introducing parallel and congruence properties into a join system. Although there is no existence theorem for a best path, we might try to search for a treatment that will be as satisfactory to us as Euclid's must have appeared to his contemporaries.

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* See [2], Chapter VIII for a discussion of descriptive geometry and [10] for the relation between descriptive geometries and join systems.

APPENDIX. EXAMPLES OF JOIN SYSTEMS

In this appendix examples and modes of construction of join systems are presented, in order to give body to our theory and to serve as independence examples for the exchange, comparability and separation properties of a join system.

A. The arithmetic affine plane. Let R_2 be the set of all ordered pairs (x_1, x_2) of real numbers. The elements of R_2 may be called points or vectors. Using vectorial shorthand, we write x for (x_1, x_2) and define addition and subtraction in R_2, and multiplication of elements of R_2 by real numbers, as follows:

$$x \pm y = (x_1, x_2) \pm (y_1, y_2) = (x_1 \pm y_1, x_2 \pm y_2),$$
$$\lambda x = \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2).$$

In R_2 we define join in this way: ab is the set of all elements x of R_2 expressible in the form $x = \lambda a + \mu b$, where λ, μ are real numbers which satisfy $0 < \lambda, \mu$; $\lambda + \mu = 1$. This definition is suggested by the familiar point of division formula of elementary analytic geometry. We are saying in effect that the join of a and b is to be the set of points "between" a and b, which is the set of points that divide "segment ab" in a positive ratio.

R_2, with the given definition of join, is a join system.

To verify (J1) observe first that $ab \subset R_2$ by definition of join, and second that $ab \neq \varnothing$ since it contains, for example, $\frac{1}{2}a + \frac{1}{2}b$. (J2) and the first part of (J6) are easily seen to be valid in R_2.

Verification of (J3). First we show that

(A.1) $$x \subset (ab)c$$

holds if and only if x is expressible in the form

(A.2) $$x = \lambda a + \mu b + \nu c, \qquad 0 < \lambda, \mu, \nu, \qquad \lambda + \mu + \nu = 1.$$

Suppose (A.1). Then

(A.3) $$x \subset yc, \qquad y \subset ab$$

holds for some y. By definition of join, (A.3) yields

(A.4) $$x = \alpha y + \beta c, \qquad 0 < \alpha, \beta, \qquad \alpha + \beta = 1;$$

(A.5) $$y = \gamma a + \delta b, \qquad 0 < \gamma, \delta, \qquad \gamma + \delta = 1.$$

Eliminating y between (A.4), (A.5) and determining λ, μ, ν by

(A.6) $$\lambda = \alpha\gamma, \qquad \mu = \alpha\delta, \qquad \nu = \beta,$$

we obtain (A.2).

Conversely suppose that x satisfies (A.2). We retrace our steps to (A.1). First we determine α, β, γ, δ to satisfy (A.6) and the relation $\alpha + \beta = 1$. We get

$$\alpha = 1 - \nu = \lambda + \mu, \qquad \beta = \nu, \qquad \gamma = \lambda/(\lambda + \mu), \qquad \delta = \mu/(\lambda + \mu).$$

Then $0<\alpha, \beta, \gamma, \delta$ and $\gamma+\delta=1$. Further $x=\alpha(\gamma a+\delta b)+\beta c$. Determine y by (A.5), so that (A.4) follows. Relations (A.4), (A.5) imply (A.3) and (A.1) follows.

Similarly we show that $x\subset a(bc)$ is equivalent to (A.2). Thus $x\subset(ab)c$ is equivalent to $x\subset a(bc)$ and (J3) is verified.

(J4) is easily verified in R_2. By definition $a/b\subset R_2$, and a/b is nonvoid since it contains, for example, $2a-b$. To see this, note $a=\frac{1}{2}(2a-b)+\frac{1}{2}b$, $0<\frac{1}{2}$.

Verification of (J5). Suppose in R_2, $a/b\approx c/d$. Then for some element x of R_2 $x\approx a/b$, $x\approx c/d$. Hence $a\approx bx$, $c\approx dx$, which imply

$$a = \lambda b + \mu x, \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1, \tag{A.7}$$

$$c = \lambda' d + \mu' x, \qquad 0 < \lambda', \mu', \qquad \lambda' + \mu' = 1. \tag{A.8}$$

Eliminating x between (A.7), (A.8) we have

$$\mu' a + \mu\lambda' d = \mu'\lambda b + \mu c. \tag{A.9}$$

Further

$$\mu' + \mu\lambda' = \mu' + \mu(1 - \mu') = \mu'(1 - \mu) + \mu = \mu'\lambda + \mu.$$

Thus (A.9) implies

$$\frac{\mu' a + \mu\lambda' d}{\mu' + \mu\lambda'} = \frac{\mu'\lambda b + \mu c}{\mu'\lambda + \mu}. \tag{A.10}$$

Let y denote either member of (A.10). By the definition of join, $y\subset ad$ and $y\subset bc$, so that $ad\approx bc$ and (J5) is verified.

It remains only to verify the idempotent law, $a/a=a$. Suppose $x\subset a/a$. Then $ax\supset a$ and

$$a = \lambda a + \mu x, \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1.$$

Solving for x we have $x=\mu^{-1}(1-\lambda)a=a$, and we infer $a/a=a$.

Thus we have proved that R_2 satisfies (J1), $\cdots$, (J6) and is a join system. R_2, as a system of points and lines, is an ordered affine plane—we have in a natural manner converted R_2 into a join system, which might be called an affine planar join system.

Furthermore R_2 satisfies (E), the Exchange Postulate of Section 14 and (C), the Comparability Postulate of Section 15. In order to show this we introduce two lemmas which are of some interest in themselves.

LEMMA 1. *a/b is the set of all elements x of R_2 expressible in the form*

$$x = \lambda a + \mu b, \qquad \mu < 0, \qquad \lambda + \mu = 1. \tag{A.11}$$

Proof. Let $x\subset a/b$. Then $a\subset bx$ so that

$$a = \rho b + \sigma x, \qquad 0 < \rho, \sigma, \qquad \rho + \sigma = 1$$

Solving for x we get $x=\sigma^{-1}a-\sigma^{-1}\rho b$. Let $\lambda=\sigma^{-1}$, $\mu=-\sigma^{-1}\rho$. Then $\mu<0$ and $\lambda+\mu=1$ so that (A.11) is valid. Conversely, it is easily shown that (A.11) implies $x\subset a/b$, and the lemma holds.

LEMMA 2. *$\{a, b\}$ is the set of all elements x of R_2 expressible in the form*

$$x = \lambda a + \mu b, \qquad \lambda + \mu = 1. \tag{A.12}$$

Proof. Let $x\subset\{a, b\}$. Since R_2 is a join system we have $\{a, b\}=ab/ab$ by Theorem 16, Corollary. Thus $x\subset ab/ab$ so that $x\subset p/q$ where $p, q\subset ab$. Lemma 1 implies

$$x = \rho p + \sigma q, \qquad \rho + \sigma = 1; \tag{A.13}$$

and the definition of join in R_2 yields

$$p = \alpha a + \beta b, \qquad \alpha + \beta = 1, \tag{A.14}$$

$$q = \gamma a + \delta b, \qquad \gamma + \delta = 1. \tag{A.15}$$

Eliminating p, q between (A.13), (A.14), (A.15) and letting $\lambda=\rho\alpha+\sigma\gamma$, $\mu=\rho\beta+\sigma\delta$ we obtain (A.12).

Conversely suppose (A.12). If $\lambda=0$ or $\mu=0$ then $x=b$ or $x=a$ and $x\subset\{a, b\}$. Suppose $\lambda, \mu\neq 0$. Then we have

$$\mu < 0, \qquad \lambda < 0, \quad \text{or} \quad 0 < \lambda, \mu. \tag{A.16}$$

This implies, by Lemma 1 and the definition of join in R_2,

$$x \subset a/b \cup b/a \cup ab \subset \{a, b\},$$

and the lemma is established.

Verification of Postulate (E). Suppose that

$$b \subset \{a_1, a_2\} \tag{A.17}$$

holds, where $b\neq a_2$. We show that $\{a_1, a_2\}=\{b, a_2\}$. By definition of linear closure of a set (Sec. 9), relation (A.17) implies $\{b, a_2\}\subset\{a_1, a_2\}$. To establish the reverse inclusion we apply Lemma 2 to (A.17), obtaining

$$b = \lambda a_1 + \mu a_2, \qquad \lambda + \mu = 1.$$

Here $\lambda\neq 0$ since $b\neq a_2$. Thus we can solve for a_1, getting $a_1=\lambda^{-1}b-\lambda^{-1}\mu a_2$. Lemma 2 applies and yields $a_1\subset\{b, a_2\}$. Thus $\{a_1, a_2\}\subset\{b, a_2\}$ and we conclude that $\{a_1, a_2\}=\{b, a_2\}$.

Verification of Postulate (C). Suppose that $x\subset\{a, b\}$. By Lemma 2, $x=\lambda a+\mu b$, where $\lambda+\mu=1$, that is, (A.12) holds. Then by the argument in the second paragraph of the proof of Lemma 2

$$x \subset a/b \cup ab \cup b/a \cup a \cup b.$$

Thus

$$\{a, b\} \subset a/b \cup ab \cup b/a \cup a \cup b.$$

The reverse inclusion is always valid ((9.1) of Sec. 9) so that Postulate (C) holds in R_2.

Having verified (C) in the join system R_2, we could immediately infer that (E) holds by Theorem 40, Corollary 2. We have verified (E) independently, in order to make the appendix less dependent on the body of the paper and because our proof is valid for certain join systems in which (C) fails (see Sec. G below).

Observe that all the results of Sections 14, 15, 16 on join systems that satisfy (E) or (C) are valid for R_2.

In order to find the significance of independence in R_2 and to determine its dimension, it is most convenient to have a formula for $\{a_1, \cdots, a_r\}$ in R_2. We generalize Lemma 2 in the following theorem.

THEOREM A. *$\{a_1, \cdots, a_r\}$ is the set of all elements x of R_2 expressible in the form*

$$x = \lambda_1 a_1 + \cdots + \lambda_r a_r, \qquad \lambda_1 + \cdots + \lambda_r = 1. \tag{A.18}$$

Proof. Let $S(a_1, \cdots, a_r)$ be the set of x which satisfy (A.18). First we establish, using induction,

$$\{a_1, \cdots, a_r\} \supset S(a_1, \cdots, a_r). \tag{A.19}$$

Relation (A.19) is valid for $r=1$ (and for $r=2$ by Lemma 2). Suppose (A.19) is valid for $r=k$. Let $x \subset S(a_1, \cdots, a_{k+1})$, which implies

$$x = \lambda_1 a_1 + \cdots + \lambda_k a_k + \lambda_{k+1} a_{k+1}, \qquad \lambda_1 + \cdots + \lambda_{k+1} = 1. \tag{A.20}$$

At least one of the λ's is not 1. It is not restrictive to suppose $\lambda_{k+1} \neq 1$. Then $\lambda = \lambda_1 + \cdots + \lambda_k \neq 0$ and (A.20) implies

$$x = \lambda x' + \lambda_{k+1} a_{k+1}, \tag{A.21}$$

where $x' = \lambda^{-1}(\lambda_1 a_1 + \cdots + \lambda_k a_k)$. By definition of $S(a_1, \cdots, a_k)$ and the induction supposition,

$$x' \subset S(a_1, \cdots, a_k) \subset \{a_1, \cdots, a_k\}.$$

By Lemma 2, (A.21) implies

$$x \subset \{x', a_{k+1}\} \subset \{a_1, \cdots, a_k, a_{k+1}\},$$

so that $\{a_1, \cdots, a_{k+1}\} \supset S(a_1, \cdots, a_{k+1})$, and (A.19) holds for $r=k+1$. Thus (A.19) is valid by induction.

Now we show that $S(a_1, \cdots, a_r)$ is linear. Suppose that $p, q \subset S(a_1, \cdots, a_r)$, and $x \subset \{p, q\}$. A straightforward algebraic argument using Lemma 2 shows that $x \subset S(a_1, \cdots, a_r)$. Thus

$$S(a_1, \cdots, a_r) \supset \{p, q\} \supset pq, p/q$$

and $S(a_1, \cdots, a_r)$ is linear by definition.

Finally note that $S(a_1, \cdots, a_r) \supset a_1, \cdots, a_r$. Thus

$$S(a_1, \cdots, a_r) \supset \{a_1, \cdots, a_r\}$$

by definition of $\{a_1, \cdots, a_r\}$, and the theorem is established.

Now we obtain an algebraic criterion for independence of elements of R_2. The criterion involves the vector "zero," that is the ordered pair (0, 0), which we denote by 0. The context will always indicate whether 0 denotes the real number zero, or the vector zero of R_2.

THEOREM B. *$a_1, \cdots, a_r$ are independent, if and only if, the relations*

$$\lambda_1 a_1 + \cdots + \lambda_r a_r = 0, \qquad \lambda_1 + \cdots + \lambda_r = 0 \tag{A.22}$$

imply for all choices of $\lambda_1, \cdots, \lambda_r$

$$\lambda_1 = \cdots = \lambda_r = 0. \tag{A.23}$$

Proof. Suppose $a_1, \cdots, a_r$ are not independent. We show that (A.22) does not imply (A.23) for certain $\lambda_1, \cdots, \lambda_r$. By Theorem 36, Corollary 4

$$a_i \subset \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r\}$$

for some a_i. By the last theorem we have

$$a_i = \lambda_1 a_1 + \cdots + \lambda_{i-1} a_{i-1} + \lambda_{i+1} a_{i+1} + \cdots + \lambda_r a_r,$$

where $\lambda_1 + \cdots + \lambda_{i-1} + \lambda_{i+1} + \cdots + \lambda_r = 1$. Letting $\lambda_i = -1$, these yield

$$\lambda_1 a_1 + \cdots + \lambda_r a_r = 0, \qquad \lambda_1 + \cdots + \lambda_r = 0.$$

Thus (A.22) holds but (A.23) fails.

Conversely suppose (A.22) does not always imply (A.23). Then there exist numbers $\lambda_1, \cdots, \lambda_r$ not all zero which satisfy (A.22) but not (A.23). Suppose $\lambda_i \neq 0$. Then we can solve (A.22) for a_i in terms of the other a's, and apply the last theorem to yield

$$a_i \subset \{a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r\}.$$

Thus by Theorem 36, Corollary 4, $a_1, \cdots, a_r$ are not independent, and our proof is complete.

Now we can show that R_2, as a join system, has dimension 3. Let $e^0 = (0, 0)$, $e^1 = (1, 0)$, $e^2 = (0, 1)$. We show that e^0, e^1, e^2 form a basis (Sec. 14) of R_2. Suppose that

$$\lambda_0 e^0 + \lambda_1 e^1 + \lambda_2 e^2 = 0, \qquad \lambda_0 + \lambda_1 + \lambda_2 = 0.$$

Then $(\lambda_1, \lambda_2) = 0 = (0, 0)$ so that $\lambda_1 = \lambda_2 = 0$. Clearly $\lambda_0 = 0$, and e^0, e^1, e^2 are independent by the last theorem. Further by Theorem A, $\{e^0, e^1, e^2\}$ is the set of elements of the form $\lambda_0 e^0 + \lambda_1 e^1 + \lambda_2 e^2 = (\lambda_1, \lambda_2)$ where $\lambda_0 + \lambda_1 + \lambda_2 = 1$. Thus

$\{e^0, e^1, e^2\} = R_2$ so that e^0, e^1, e^2 form a basis of R_2, and the dimension of R_2 is 3.

B. Generalization to affine n-space. The discussion above can be generalized in several ways. First we can extend it, with little trouble, from ordered pairs to ordered n-tuples of real numbers.

Let R_n be the set of ordered n-tuples $(x_1, \cdots, x_n)$ of real numbers. As in R_2 we write x for $(x_1, \cdots, x_n)$ and define

$$x \pm y = (x_1, \cdots, x_n) \pm (y_1, \cdots, y_n) = (x_1 \pm y_1, \cdots, x_n \pm y_n),$$
$$\lambda x = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n).$$

The element $(0, \cdots, 0)$ of R_n is called the *zero* of R_n and is denoted by 0.

We define join in R_n exactly as in R_2. Then the theory presented in Section A holds just as well for R_n as for R_2 (except for the discussion in the last paragraph of Sec. A on the dimension of R_2). This is not very remarkable, since the algebraic properties employed are independent of the number of components and so are equally valid in R_n and R_2. In fact, if you are familiar with the notion of vector space, you may observe that the discussion of Section A (except for the part on dimension) is applicable to any vector space over the real field (or an ordered field). Thus the definition of join of Section A serves to convert any such vector space into a join system.

To determine the dimension of R_n, define $e^0 = (0, \cdots, 0)$, $e^1 = (1, 0, \cdots, 0)$, $\cdots$, $e^n = (0, \cdots, 0, 1)$ and show, using Theorems A, B that $e^0, e^1, \cdots, e^n$ form a basis of R_n.

Summary. R_n is a join system that satisfies Postulates (E) and (C) and has dimension $n+1$.

C. An infinite-dimensional join system. The restriction to n components in the above discussion is not essential. We can easily extend it to "vectors" with infinitely many components.

Let R^* be the set of all infinite sequences $(x_1, \cdots, x_n, \cdots)$ of real numbers. We write x for $(x_1, \cdots, x_n, \cdots)$ and define

$$x \pm y = (x_1, \cdots, x_n, \cdots) \pm (y_1, \cdots, y_n, \cdots) = (x_1 \pm y_1, \cdots, x_n \pm y_n, \cdots),$$
$$\lambda x = \lambda(x_1, \cdots, x_n, \cdots) = (\lambda x_1, \cdots, \lambda x_n, \cdots).$$

The infinite sequence $(0, \cdots, 0, \cdots)$ is called the *zero* of R^* and denoted by 0.

We define join in R^* as in R_2. The discussion of Section A, being vector algebraic in character, is still independent of the number of components and is applicable to R^*. Thus R^*, with join as defined, is a join system that satisfies Postulates (E) and (C). However unlike R_n, R^* is infinite dimensional (in the sense of Sec. 14, last paragraph).

To show this, let $e^0 = (0, \cdots, 0, \cdots)$ and e^r $(r = 1, 2, \cdots)$ be the element of R^* in which the rth term is 1 and the other terms are 0. By Theorem B, $e^0, e^1, \cdots, e^{n-1}$ are independent for each n. Thus R^* contains n independent

elements for each natural number n—consequently it has no finite basis and is infinite dimensional.

D. The direct product of two join systems. We present in this section a formal procedure for constructing a join system from two given ones.

Let $(G_1, \cdot)$, $(G_2, \cdot)$ be any two join systems—for the sake of convenience we use the same symbol to denote their operations. If $A_1 \subset G_1$, $A_2 \subset G_2$ we use the symbol (A_1, A_2) to denote the cartesian product of A_1 and A_2, that is the set of ordered pairs (a_1, a_2) where $a_1 \subset A_1$, $a_2 \subset A_2$. Let $G = (G_1, G_2)$, the cartesian product of G_1 and G_2. We define a join operation $\cdot$ in G as follows:

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot b_1, a_2 \cdot b_2).$$

We call G with join as defined the *direct product** of the join systems G_1, G_2. We show that G is a join system.

In G the operation $\cdot$ is extended to sets and the inverse operation / is defined in the usual way. Then we can prove

$$(A_1, A_2) \cdot (B_1, B_2) = (A_1 \cdot B_1, A_2 \cdot B_2), \tag{D.1}$$

$$(a_1, a_2)/(b_1, b_2) = (a_1/b_1, a_2/b_2). \tag{D.2}$$

We justify (D.2). Suppose

$$(x_1, x_2) \subset (a_1, a_2)/(b_1, b_2). \tag{D.3}$$

By definition of /, $(b_1, b_2) \cdot (x_1, x_2) \supset (a_1, a_2)$, so that by definition of join in G $(b_1 \cdot x_1, b_2 \cdot x_2) \supset (a_1, a_2)$. This implies, by definition of cartesian product,

$$b_1 \cdot x_1 \supset a_1, \qquad b_2 \cdot x_2 \supset a_2.$$

Thus $x_1 \subset a_1/b_1$, $x_2 \subset a_2/b_2$ and we infer, again using the definition of cartesian product,

$$(x_1, x_2) \subset (a_1/b_1, a_2/b_2). \tag{D.4}$$

Conversely, supposing (D.4), we can retrace the steps to (D.3) so that (D.2) is proved.

Using (D.1), (D.2) it is not difficult to show that (J1), $\cdots$, (J6) hold in G, since the operations which occur in (J1), $\cdots$, (J6) may be performed component by component. As an illustration we show that (J5) holds in G. Suppose that

$$(a_1, a_2)/(b_1, b_2) \approx (c_1, c_2)/(d_1, d_2).$$

By (D.2) $(a_1/b_1, a_2/b_2) \approx (c_1/d_1, c_2/d_2)$ so that $a_1/b_1 \approx c_1/d_1$, $a_2/b_2 \approx c_2/d_2$. Since (J5) is valid in G_1 and G_2, $a_1 d_1 \approx b_1 c_1$, $a_2 d_2 \approx b_2 c_2$. Thus

* Compare the idea of direct product in group theory; see, for example, K. S. Miller, Elements of Modern Abstract Algebra, New York, 1958.

$$(a_1d_1, a_2d_2) \approx (b_1c_1, b_2c_2), \qquad (a_1, a_2)\cdot(d_1, d_2) \approx (b_1, b_2)\cdot(c_1, c_2)$$

and (J5) is valid in G.

E. The direct product of two lines. We consider now an interesting case of the direct product of two join systems which serves as an important counter-example.

Let G_1 be the real number system and let a join operation be defined in G_1 as follows: $a\cdot b$, if $a\neq b$, is the set of real numbers between a and b; and $a\cdot a=a$. It is not hard to show that G_1, with $\cdot$ as defined, is a join system—in fact it is equivalent to the join system R_n of Section B when n is taken to be 1.

Now let G be the direct product of G_1 and G_1. Note that G may be considered to be the cartesian plane with an unusual kind of join operation. G is a join system which does not satisfy the Exchange Postulate (E) of Section 14—its existence attests that (E) and the familiar Euclidean theory of incidence (Sec. 14) are independent of (J1), $\cdots$, (J6).

We show that Postulate (E) fails in G. Let x denote the arbitrary element (x_1, x_2) of G. Choose $a=(a_1, a_2)$ and $b=(b_1, b_2)$ such that $a_1<b_1$, $a_2<b_2$ (see Fig. 25). Then $a\cdot b$, the join of a and b in G, consists of all x such that

$$a_1 < x_1 < b_1, \qquad a_2 < x_2 < b_2.$$

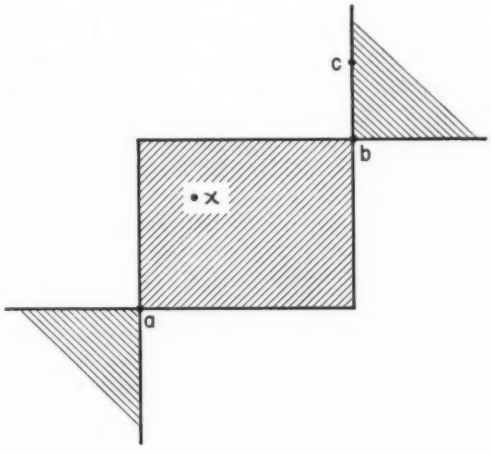

FIG. 25

Thus $a\cdot b$ is represented by the interior of the rectangle which has horizontal and vertical sides and a, b as a pair of opposite vertices (Fig. 25). Furthermore $\{a, b\}$ is seen to contain all the elements of G, and is represented by the cartesian plane (Fig. 25).

Now choose c so that $c_1=b_1$, $c_2\neq b_2$ (Fig. 25). We have

(E.1) $$c \subset \{a, b\}, \qquad c \neq b.$$

Observe that $\{c, b\}$ consists of all y ($y = (y_1, y_2)$) such that $y_1 = b_1$, and is represented by the vertical cartesian line determined by b and c. Thus $\{a, b\} \neq \{c, b\}$. This with (E.1) shows that Postulate (E) is false in G.

It immediately follows that the Comparability Postulate (C) of Section 15 fails in G, since Postulate (C) implies Postulate (E) in any join system (Th. 40, Corol. 2).

Furthermore the join system G serves to indicate the restrictiveness in the conventional definition of line, since in G, $a/b \cup ab \cup b/a \cup a \cup b$ is neither a linear nor a convex set (see shaded area in Fig. 25). Note also that the elements a, b, c of G do not satisfy the Comparability Property of Section 15—for they are distinct and collinear but neither of the three is between the other two, since neither is in the join of the other two. This suggests the possibility of a theory of "partially ordered" join systems (or geometries) in which the Comparability Property fails (see Sec. G).

F. Subsystems of a join system. We show now how to construct a join system from a given one by using a suitable type of convex subset. The procedure is similar to that employed at the end of Section 8 to convert a linear subset of a join system into a join system.

Let K be a convex set of a join system G. Since K is closed under join it is natural to ask about its properties relative to the join operation restricted in its application to the elements of K. In particular is K (relative to this operation) a join system? To answer this observe that (J1) holds in K since K is a subset of G which is closed under join. (J2), (J3) and the idempotent law $aa = a$ hold in K since they are formal laws valid for any elements of G. The transposition law (J5), that $a/b \approx c/d$ implies $ad \approx bc$, requires a bit more thought for two reasons: first, it involves the defined operation extension, and the extension operation in K need not coincide with that in G; second, it is not simply a formula since its hypothesis and conclusion are existential statements. However we can express (J5) equivalently as: If $a \approx bx$ and $c \approx dx$ for some x then $ad \approx bc$. This is valid when the elements are restricted to K, since it is valid in G and K is a convex set in G. Similarly the idempotent law $a/a = a$ can be rewritten, $ax \approx a$ if and only if $x = a$, and holds in K.

Summary. (J1), (J2), (J3), (J5), and (J6) are valid in K.

(J4) need not hold in K. Counterexamples are easily found. For example, in a Euclidean geometry (considered as a join system), take K as a closed interval with endpoints p, q or a closed triangular region with vertices p, q, r. In each case (J4) fails, since the extension operation in K satisfies $p/q = \varnothing$. Note however that a segment (open interval) or a triangle interior does satisfy (J4). Thus we have an answer to our question concerning the system K; K is a join system if and only if it satisfies (J4).

Let us examine this more closely. Actually all we require of (J4) is the condition $a/b \neq \varnothing$, since $a/b \subset K$ by definition. Stated a bit more precisely the condition is: If $a, b \subset K$ then $a/b \approx K$ or, equivalently, $a \approx bK$. That is, $a \subset K$ implies $a \subset bK$ for $b \subset K$. This is equivalent to: $K \subset xK$ for $x \subset K$. This is an important property of a convex set which we signalize in the

DEFINITION. *Let K be a convex set in join system G such that $K \subset xK$ for $x \subset K$. Then we say K is an* ***open convex set*** *or simply K is* ***open.***

We summarize our discussion more formally and perspicuously by introducing the term subsystem.

DEFINITION. *Let A be a subset of join system G which is a join system relative to the join operation of G restricted to the elements of A. Then A is a* ***subsystem*** *(or a sub-join-system) of G.*

Then we may assert the

THEOREM. *Let K be a convex subset of join system G. Then K is a subsystem of G if and only if K is an open convex set.*

Thus every open convex subset of G gives rise to an "induced" join system or subsystem in a natural manner. Can nonconvex sets behave similarly? The answer is no—since if A is a subsystem of G, it must, by definition, satisfy Postulate (J1) and be convex. It is interesting to compare our results with conditions in group theory that a subset be a subgroup.

The theorem yields a new mode of construction of join systems provided we can "find" the open convex sets of a given join system. We show now how to construct the open convex sets of a join system from its convex sets. The Euclidean examples mentioned above suggest the conjecture that if we delete the "boundary" of a convex set, we get an open convex set. We proceed to clarify and justify this for any join system.

Suppose a convex set K is not open—that is, $K \subset xK$ for all $x \subset K$ is false. Then there exist $a, b \subset K$ such that $a \subset bK$ is false, or equivalently $a/b \approx K$ is false. In Euclidean geometry, this means that ray a/b is disjoint to K and consequently that there are points not in K which are as close to a as we please. Thus in the Euclidean case, a would be a boundary point of K. This suggests the

DEFINITION. *Let K be a convex set in a join system and $a \subset K$. Suppose $a \not\subset bK$ (or, equivalently, $a/b \approx K$ is false) for some $b \subset K$. Then we call a, a* ***boundary*** *element of K. In the contrary case a is an* ***interior*** *element of K. That is, a is an interior element of K if $a \subset xK$ for $x \subset K$. The* ***boundary (interior)*** *of K is the set of its boundary (interior) elements.*

Comparing this definition with that of open convex set we note that a convex set is open if and only if it is identical with its interior, or equivalently its boundary is void.

By the definition the interior of convex set K is merely K "minus" its boundary, so we may state our conjecture as follows:

THEOREM. *The interior of a convex set is an open convex set.*

To prove this we introduce the

LEMMA. *Let I be the interior of convex set K. Then $a \subset I$ implies $aK = I$.*

Proof. Let $p \subset aK$. Suppose that $x \subset K$; then by definition of I, $a \subset xK$. Thus

$$p \subset aK \subset xKK = xK$$

and $p \subset I$. Conversely, $p \subset I$ implies $p \subset aK$ by definition of I.

Proof of Theorem. Let K be a convex set and I its interior. The convexity of I follows directly from the lemma, since a, $b \subset I$ implies $ab \subset aK = I$. To show that I is open, let $x \subset I$. Then we have, applying the lemma twice, $I = xK = xxK = xI$ and I is open by definition.

A final remark. Experience with convex sets in Euclidean geometries suggests that a nonvoid convex set has a nonvoid interior. This is not true for all join systems. As an example consider R^*, the infinite dimensional join system discussed in Section C, and let K be the set of infinite sequences of real numbers with no negative terms and only a finite number of positive ones. K is easily shown to be convex. But each element a of K is a boundary element. For if $a = (a_1, \cdots, a_m, 0, 0, \cdots)$ and we let $b = (b_1, \cdots, b_m, 1, 0, 0, \cdots)$ we can show that $a \not\subset bK$.

G. Join systems over a partially ordered field. In Section A we constructed a join system R_2 using algebraic and order properties of the real number system. This was generalized in Section B to a join system of arbitrary finite dimension and in Section C to one of infinite dimension. Now we develop a very different type of generalization: We alter the base field rather than the number of components. The proof in Section A that R_2 is a join system did not employ all the basic order properties of the real numbers. For example the continuity property was not needed. Our treatment in Section A would have worked just as well if we had taken the rational number system as our base field. Evidently we need merely a field satisfying certain familiar order properties in order to obtain the results of Sections A, B, C. There is a well-known concept of *ordered field* (see [1], pp. 48, 8) which is a generalization of the rational and real number systems and our treatment in Sections A, B, C holds without change if we replace R, the real number field, by any ordered field. However the notion of ordered field has been generalized to a concept of *partially ordered field*, over which join systems can be constructed by the method of Section A. Thus we shall introduce the notion of a partially ordered field and show that the construction of Section A yields a join system when applied to any partially ordered field—the discussion automatically covers ordered fields. However a join system over a partially

ordered field need not satisfy all the properties of R_2. We shall give an example of one which satisfies Postulate (E) but not Postulate (C). This is an interesting join system—its points and lines form an affine geometry but neither (C) nor Pasch's Postulate (Section 16) holds. It is thus an independence example for these principles relative to the postulate set composed of (J1), $\cdots$, (J6) and (E).

We introduced the notion of a partially ordered field in [11] with the object of constructing a "partially ordered geometry"—this may be described essentially as a join system which satisfies (E) but not (C). A partially ordered field is defined there to be a field in which a 3-term relation (betweenness) is assigned, which satisfies postulates suggested by certain betweenness properties of the real numbers. Recently Dubois has introduced a definition of partially ordered field* based on the notion of positive element. His formulation is quite simple and lends itself to our present purpose. We state it as follows:

DEFINITION. *A partially ordered field is a field with a prescribed subset P, called the set of* ***positive*** *elements, which satisfies the following postulates:*

(O1). *P is nonvoid.*

(O2). *P is closed under addition, multiplication and the operation of forming reciprocals.*

A partially ordered field F is called ***fully ordered*** *or simply an* ***ordered field*** *if it satisfies the trichotomy law:*

(O3). *If a is an element of F just one of the following alternatives holds: a is an element of P, $a=0$, or $-a$ is an element of P.*

We quickly sketch the elementary properties of partially ordered fields. We begin as in the theory of ordered fields (or ordered integral domains) (see [1], pp. 8, 9), by defining "less than" and deriving its elementary properties which are independent of trichotomy. We define $a<b$ (or $b>a$) to mean $b-a$ is positive, that is $b-a$ is in P. If $a<0$ we say a is *negative*. Note that a is positive if and only if $0<a$, or equivalently $-a$ is negative. The relation $<$ has the following properties: transitiveness, $a<b$ and $b<c$ imply $a<c$; asymmetry, $a<b$ implies $b<a$ false; irreflexiveness, $a<a$ is false. The monotonic laws hold: $a<b$ implies (1) $a+x<b+x$ for arbitrary x; (2) $ax<bx$ for positive x; and (3) $bx<ax$ for negative x. The familiar multiplicative laws for positive and negative numbers are valid: The product of two positive or two negative numbers is positive, and the product of a positive and a negative is negative.

Similar properties hold for quotients. First observe that Postulate (O2) implies P is closed under division. Then if b is negative, so is its reciprocal. For $b<0$ implies $-b$ is positive, so that its reciprocal $1/(-b)=-1/b$ is positive, and $1/b$ is negative. Then we can assert that the quotient of two positive or two nega-

* See D. W. Dubois, On partly ordered fields, Proc. Amer. Math. Soc., vol. 7, 1956, pp. 918–930. I suspect that the two notions of partially ordered field are equivalent but I have not yet proved this.

tive numbers is positive, and the quotient of a positive by a negative or a negative by a positive is negative.

For an ordered field the trichotomy law can be extended ([1], p. 8) to the form: Exactly one of the statements $a<b$, $a=b$, $b<a$ holds.

Now we indicate that any partially ordered field F has a subfield isomorphic to the rational number field. We begin with the order relations of the elements 0, 1 of F. By Postulate (O1), there is an element a in P. By Postulate (O2), P contains $a \div a = 1$. Thus $0<1$. Adding 1 to both sides we get $1<2$. Similarly $2<3$ and so on. That is we have

$$0 < 1 < 2 < 3 < \cdots .$$

Thus the positive integral elements of F are ordered in the usual way. It follows that the integral elements of F form an infinite set (F has zero characteristic) and that they behave (as regards $+$, $\times$, $<$) exactly like the ordinary signed integers. Furthermore the *rational elements* of F (quotients of integral elements) behave like the rational numbers—that is, they form a subsystem of F isomorphic to the ordered field of rational numbers. Note in particular that the rational elements of F are ordered in the familiar way, for example, in F, $0<1/2<1$.

Now let F be any partially ordered field. We form F_2 the set of ordered pairs (x_1, x_2) of elements of F. We define join in F_2 exactly as in R_2 (Sec. A). We assert that F_2 is a join system. To justify this, merely reexamine the proof that R_2 is a join system in Section A and observe that the order properties of real numbers employed are valid in any partially ordered field. Nowhere in the verification of (J1), $\cdots$, (J6) did we assume trichotomy. For example in verifying (J3) and (J5) we employed the property that the sum, product and quotient of positive numbers is positive; in the verification of (J4) we used $0<1/2$.

Furthermore the proof of Lemma 1 of Section A holds for F_2; that of Lemma 2 fails however since the trichotomy principle is required to establish relation (A.16). Indeed Lemma 2 does not hold for all partially ordered fields F. As a counterexample, we can take the complex number field and convert it into a partially ordered field by specifying P to be the set of positive real numbers.

Thus Sections A, B, C are not completely valid when we replace the real number system by an arbitrary partially ordered field. However it is easily verified that the theory is valid for any *ordered* field. In particular if F is an ordered field, the join system F_2 satisfies Postulates (E) and (C).

Now we come to our main problem: To construct a join system which satisfies (E) and not (C). Our procedure is to construct a partially ordered field G which is not fully ordered, but such that Lemma 2 holds for the join system G_2. Then we employ Lemmas 1 and 2 to establish Postulate (E) for G_2 and show that Postulate (C) fails because G is not fully ordered.

Let G be the field of rational functions in a real variable y with real coefficients, and let the set P of positive elements be composed of the rational functions other than zero which assume no negative value. G, with P as assigned, is a partially ordered field. In G, $a<b$ means that $a \neq b$ and $a(y) \leqq b(y)$ for all y

for which a and b are defined. G is not fully ordered since the identity function θ defined by $\theta(y)=y$ satisfies none of the conditions, $\theta<0$, $\theta=0$, $0<\theta$. The following property of G is crucial in our discussion: For each a in G there exists b in G such that $a<b$ and $0<b$. That is each element of G is "dominated" by a positive element. To show this observe that $a(y)<1+a^2(y)$.

We recall that Lemma 1 of Section A holds for G_2, since it is valid for any partially ordered field. Next we prove Lemma 2 valid for G_2. We must show that given elements a, b of G_2, $\{a, b\}$ is the set S of all x of G_2 which satisfy

$$x = \lambda a + \mu b, \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1, \tag{G.1}$$

where λ, μ are in the base field G. Reviewing the proof of Lemma 2, Section A as given for the real field R, we see that the first part is valid for any partially ordered field. Thus $\{a, b\} \subset S$, and we have merely to show the reverse inclusion.

Suppose then x is in S, that is, x satisfies (G.1). We introduce auxiliary elements c, d to relate x to a and b. Let λ' satisfy: $\lambda<\lambda'$, $1<\lambda'$. We determine c, d by

$$c = \lambda' a + \mu' b, \qquad \mu' = 1 - \lambda'; \qquad d = (\lambda' + 1)a + (\mu' - 1)b.$$

A formal calculation verifies that $x=(1+\lambda'-\lambda)c+(\lambda-\lambda')d$. Here $\lambda-\lambda'<0$ and Lemma 1 implies $x \subset c/d$. Further $\mu'<0$ so that $c \subset a/b$ by Lemma 1. Similarly $d \subset a/b$. Thus $x \subset \{a, b\}$ and $S \subset \{a, b\}$. This completes the verification of Lemma 2 for G_2.

It is immediate that G_2 satisfies Postulate (E), for its verification in Section A depends merely on Lemma 2.

G_2 does not satisfy Postulate (C). To see this suppose the contrary and let a, b be chosen as the elements (1, 0), (0, 0) of G_2. Let $c=(\theta, 0)$ where θ is the identity function of G. Note that θ bears no order relation to 0 or 1. We have

$$c = \theta a + (1 - \theta)b. \tag{G.2}$$

Lemma 2 implies that c is in $\{a, b\}$. Thus by Postulate (C) $c \subset ab \cup a/b \cup b/a \cup a \cup b$. Since $c \neq a, b$ we have

$$(1)\ c \subset ab, \qquad (2)\ c \subset a/b, \quad \text{or} \quad (3)\ c \subset b/a.$$

Note that the coefficients of a and b in (G.2) are uniquely determined. In view of (G.2), (1) implies by definition of join that $0<\theta$, (2) implies by Lemma 1 that $1-\theta<0$ or $1<\theta$, and similarly (3) yields $\theta<0$. Thus G_2 cannot satisfy Postulate (C).

We conclude with some remarks concerning the join system G_2. In some respects it is a familiar kind of system. Its points and lines form an affine geometry. Actually its lines are representable as linear equations over field G just as the lines of a Euclidean plane are representable by linear equations over the real field. In fact $x=\lambda a+\mu b$, $\lambda+\mu=1$, $(a \neq b)$ is a sort of parametric equation for the line $\{a, b\}$ and we can show that

$$(a_2 - b_2)x_1 + (b_1 - a_1)x_2 + a_1 b_2 - a_2 b_1 = 0$$

is an equivalent expression for it in ordinary linear (nonparametric) form.

Moreover G_2 admits a notion of betweenness for collinear points (see the first definition of Sec. 15) which enjoys many of the familiar properties of betweenness in Euclidean geometry, but does not satisfy the Comparability Property of Section 15, that is there exist three distinct collinear points no one of which is between the other two. Thus the points of a line in G_2 need not be "fully ordered" by the betweenness relation and form, relative to it, a *partially ordered* set in the sense of [11].

A comparable situation exists in the plane, for Pasch's Postulate (Sec. 16) fails in G_2—that is, there exists a line and three distinct points not on it such that no two of the points are on the same side of the line. To show this, let line $L = \{p, q\}$, where $p = (0, 0)$, $q = (1, 0)$ and let $a = (0, 1)$, $b = (0, -1)$, $c = (1, \theta)$, where θ is the identity function of G. In view of Lemma 2, L is the set of points x $(x = (x_1, x_2))$ which satisfy $x_2 = 0$. Thus $L \not\supset a, b, c$. Similarly we show $c \not\subset \{a, b\}$ so that a, b, c are independent. Observe that the discussion at the end of Section A on the dimension of R_2 applies to G_2, since it is based on Lemmas 1 and 2 and Postulate (E). Thus G_2 has dimension 3 and $L \subset G_2 = \{a, b, c\}$. Finally L meets ab, since L contains $(0, 0)$. Thus if Pasch's Postulate is valid in G_2, L meets ac or bc. Suppose the former. Then

$$(x_1, 0) = \lambda(0, 1) + \mu(1, \theta), \qquad 0 < \lambda, \mu, \qquad \lambda + \mu = 1$$

holds for some x_1, λ, μ of G. Thus $0 = \lambda + \mu\theta$ so that $\theta = -\lambda \cdot \mu^{-1} < 0$, which is impossible. Similarly, L meets bc implies $\theta > 0$ which is equally impossible, and G_2 does not satisfy Pasch's Postulate. As is well known this implies that (plane) G_2 is not separated by all of its lines.

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